Web Appendix to "Robust Comparative Statics of Risk Changes"

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This web appendix provides supplementary proofs to the examples given in the paper "Robust comparative statics of risk changes." The proofs of the motivating example (Section 2) and of Example 1 are in the main body of the paper, so this appendix concentrates in the proof of Examples 2-6.

Example 2. Precautionary saving with time non-separable utility

Define \( X_i = \arg \max_{x \in B} EU(x, \hat{\varepsilon}_i) = \arg \max_{x \in B} Eh(y_0 - x, x + \hat{\varepsilon}_i), \) \( i = 1, 2. \) Suppose that \( \varepsilon_1 \succ_N \varepsilon_2, \) that \( h \) is \( N + 1 \) times differentiable in the second argument and once in the first argument, and that \( B \) is a compact interval of \( \mathbb{R}. \) Then, \( X_2 \geq_S X_1 \) if, equivalently, one of the following conditions hold:

- Lottery \( A = [y_0 - x^l, x^l + \hat{\varepsilon}_1; y_0 - x^h, x^h + \hat{\varepsilon}_2] \) is preferred to lottery \( B = [y_0 - x^h, x^h + \hat{\varepsilon}_1; y_0 - x^l, x^l + \hat{\varepsilon}_2] \) for all \( \varepsilon_1 \succ_N \varepsilon_2 \) and all \( x^h \geq x^l. \)
- \( (-1)^N (h^{(0,N+1)} - h^{(1,N)}) \geq 0. \)

**Proof.** Let \( x_1 \in X_1 \) and \( x_2 \in X_2. \) We show that \( \max \{x_1, x_2\} \in X_2. \) We have

\[
0 \geq Eh(y_0 - \max \{x_1, x_2\}, \max \{x_1, x_2\} + \hat{\varepsilon}_2) - Eh(y_0 - x_2, x_2 + \hat{\varepsilon}_2) \\
\geq Eh(y_0 - \max \{x_1, x_2\}, \max \{x_1, x_2\} + \hat{\varepsilon}_1) - Eh(y_0 - x_2, x_2 + \hat{\varepsilon}_1) \\
\geq 0
\]

The first inequality follows from \( x_2 \in X_2, \) the second inequality follows from the statement that lottery \( A \) is preferred to lottery \( B \) (equivalently, that the payoff function has decreasing differences in \((x, \varepsilon)\)), and the third inequality follows from the definition of \( X_1 \) and \( x_1 \in X_1. \) Since the inequalities are enclosed by zeroes, it follows that \( \max \{x_1, x_2\} \in X_2. \) Similarly, \( \min \{x_1, x_2\} \in X_1, \) so we conclude that \( X_2 \geq_S X_1. \)

By Lemma 1 in the main body of the paper, lottery \( A \) is preferred to lottery \( B \) for all \( \varepsilon_1 \succ_N \varepsilon_2 \) and all \( x^h \geq x^l \) if and only if \((-1)^N (h^{(0,N)}(y_0 - x^h, x^h + \varepsilon) - h^{(0,N)}(y_0 - x^l, x^l + \varepsilon)) \geq 0 \) for all \( \varepsilon, \) i.e. if and only if \((-1)^N (h^{(0,N+1)} - h^{(1,N)}) \geq 0, \) which is the required result. We remark that \( U^{(0,N)} = h^{(0,N)} \) and, by continuity, \( U^{(1,N)} = h^{(0,N+1)} - h^{(1,N)}. \) Thus, \((-1)^N (h^{(0,N+1)} - h^{(1,N)}) \geq 0 \iff (-1)^N U^{(1,N)} \geq 0, \) as established in Proposition 1. 

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Example 3. Saving and risky interest rates

Define $X_i = \arg \max_{x \in B} EU (x, \hat{\epsilon}_i) = \arg \max_{x \in B} u (y_0 - x) + Ev (x \hat{\epsilon}_i)$, $i = 1, 2$. Suppose that \( \hat{\epsilon}_1 \succ_n \hat{\epsilon}_2 \), that $v$ is $N + 1$ times differentiable with its successive derivatives alternating in sign, and that $B$ is a compact interval. No restrictions are imposed on $u$. Then, $X_2 \geq_x X_1$ if, equivalently, one of the following conditions hold:

- Lottery $A = [x^l \hat{\epsilon}_1; x^h \hat{\epsilon}_2]$ is preferred to lottery $B = [x^h \hat{\epsilon}_1; x^l \hat{\epsilon}_2]$ for all $\hat{\epsilon}_1 \succ_n \hat{\epsilon}_2$ and all $x^h \geq x^l$ (both lotteries defined over date-1 consumption).
- The measure of Nth degree relative risk aversion (Eeckhoudt and Schlesinger (2008)), $\frac{v^{(N+1)}(x)x}{v^{(N)}(x)}$, is no smaller than $N$.

**Proof.** Let $x_1 \in X_1$ and $x_2 \in X_2$. We show that $\max \{x_1, x_2\} \in X_2$. We have

\[
0 \geq [u (y_0 - \max \{x_1, x_2\}) + Ev (\max \{x_1, x_2\} \hat{\epsilon}_2)] - [u (y_0 - x_2) + Ev (x_2 \hat{\epsilon}_2)] \\
\geq [u (y_0 - \max \{x_1, x_2\}) + Ev (\max \{x_1, x_2\} \hat{\epsilon}_1)] - [u (y_0 - x_2) + Ev (x_2 \hat{\epsilon}_1)] \\
\geq 0
\]

The first inequality follows from $x_2 \in X_2$, the second inequality follows from the statement that lottery $A$ is preferred to lottery $B$ (equivalently, that $Ev (x \hat{\epsilon})$ has decreasing differences in $(x, \hat{\epsilon})$ and, by Remark 1 in Section 3 of the paper, that lifetime utility has decreasing differences in $(x, \epsilon)$), and the third inequality follows from the definition of $X_1$ and $x_1 \in X_1$. Since the inequalities are enclosed by zeroes, it follows that $\max \{x_1, x_2\} \in X_2$. Similarly, $\min \{x_1, x_2\} \in X_1$, so we conclude that $X_2 \geq_x X_1$.

By Lemma 1 in the main body of the paper, lottery $A$ is preferred to lottery $B$ for all $\hat{\epsilon}_1 \succ_n \hat{\epsilon}_2$ and all $x^h \geq x^l$ if and only if $(-1)^N \left( v^{(N)} (x^h \epsilon) \frac{[x^h]^N - v^{(0,N)} (x^h \epsilon)}{[x^l]^N} \right) \geq 0$ for all $\epsilon$ (i.e. $(-1)^N U^{(0,N)}$ is increasing in $x$, as in Proposition 1). Given the differentiability assumptions, this condition is equivalent to $(-1)^N \left( v^{(N+1)} (x \epsilon) x + N v^{(N)} (x \epsilon) \right) \geq 0$. With the assumption that the successive derivatives of $v$ alternate in sign, we obtain the condition in the example, $-\frac{v^{(N+1)}(x)x}{v^{(N)}(x)} \geq N$.

Example 4. Self-protection with background risk

Define $X_i = \arg \max_{x \in B} EU (x, \check{\epsilon}_i) = \arg \max_{x \in B} \{u (y_0 - x) + P (x) Ev (y_1 + \check{\epsilon}_i) + [1 - P (x)] Ev (y_1 - L + \check{\epsilon}_i)\}$, $i = 1, 2$. Suppose that $B$ is a compact interval of $\mathbb{R}$ and that $u$, $v$, and $P$ are strictly increasing and smooth. Then, if the decision maker is strictly prudent, in the sense that lottery $[y_1 - L + \check{\epsilon}_1; y_1 + \check{\epsilon}_2]$ is strictly preferred to lottery $[y_1 + \check{\epsilon}_1; y_1 - L + \check{\epsilon}_2]$ for all $\check{\epsilon}_1 \succ \check{\epsilon}_2$, and if $x^*_1 \in X_1$ and $x^*_2 \in X_2$, then $x^*_2 \geq x^*_1$.

**Proof.** Wang and Li (forthcoming) provide a proof of the statement. We present a simpler and more general proof along the lines of the discussion in our paper. Rewrite the payoff function as $f_1 (x) + Ef_2 (\check{\epsilon}_i) + P (x) Eg (\check{\epsilon}_i)$, where $g (\check{\epsilon}_i) = [v (y_1 + \check{\epsilon}_i) - v (y_1 - L + \check{\epsilon}_i)]$. Let $x^*_1 \in X_1$ and $x^*_2 \in X_2$. Suppose that $x^*_1 > x^*_2$. We have
\[ 0 \geq [f_1(x^*_1) + Ef_2(\bar{\epsilon}_2) + P(x^*_1) Eg(\bar{\epsilon}_2)] - [f_1(x^*_2) + Ef_2(\bar{\epsilon}_2) + P(x^*_2) Eg(\bar{\epsilon}_2)] \]
\[ > [f_1(x^*_1) + Ef_2(\bar{\epsilon}_1) + P(x^*_1) Eg(\bar{\epsilon}_1)] - [f_1(x^*_2) + Ef_2(\bar{\epsilon}_1) + P(x^*_2) Eg(\bar{\epsilon}_1)] \]
\[ \geq 0 \]

The first inequality follows from \( x^*_2 \in X_2 \), the second inequality follows from the assumptions of strict prudence and that \( P(x) \) is strictly increasing, and the third inequality follows from \( x^*_1 \in X_1 \). Since the inequalities are enclosed by zeroes, there is a contradiction.\[ \square \]

**Example 5. Borrowing and human capital investments**

Define \( X_i = \arg\max_{x \in B} EU(x, \bar{\epsilon}_i) = \arg\max_{(h,d) \in B} u(y - h + d) + Ev(h\bar{\epsilon}_i - d) \), \( i = 1,2 \), and \( \bar{\epsilon}_1 \succ \bar{\epsilon}_2 \). Suppose that \( B \) is a compact sublattice of \( \mathbb{R}^2 \), that \( v \) is concave and twice differentiable, and that \( u \) is concave. Then, \( X_1 \geq_S X_2 \) if, equivalently, one of the following conditions hold:

- Date-1 lottery \( A = [h'\bar{\epsilon}_1 - d', h''\bar{\epsilon}_2 - d'] \) is preferred to date-1 lottery \( B = [h''\bar{\epsilon}_1 - d'', h'\bar{\epsilon}_2 - d'] \) for all \( h' \geq h'' \), \( d' \geq d'' \), and all \( \bar{\epsilon}_1 \succ \bar{\epsilon}_2 \)

- \( v^{(2)}(h\epsilon - d)h^2 \) is decreasing in \((h,d)\).

**Proof.** Let \((h_1, d_1) \in X_1 \) and \((h_2, d_2) \in X_2 \). We need to show that \((\max\{h_1, h_2\}, \max\{d_1, d_2\}) \in X_1 \) and \((\min\{h_1, h_2\}, \min\{d_1, d_2\}) \in X_2 \). We have

\[ 0 \geq [u(y - \max\{h_1, h_2\} + \max\{d_1, d_2\}) + Ev(\max\{h_1, h_2\} \bar{\epsilon}_1 - \max\{d_1, d_2\})] - [u(y - h_1 + d_1) + Ev(h_1\bar{\epsilon}_1 - d_1)] \]
\[ \geq [u(y - \max\{h_1, h_2\} + \max\{d_1, d_2\}) + Ev(\max\{h_1, h_2\} \bar{\epsilon}_2 - \max\{d_1, d_2\})] - [u(y - h_1 + d_1) + Ev(h_1\bar{\epsilon}_2 - d_1)] \]
\[ \geq [u(y - h_2 + d_2) + Ev(h_2\bar{\epsilon}_2 - d_2)] - [u(y - \min\{h_1, h_2\} + \min\{d_1, d_2\}) + Ev(\min\{h_1, h_2\} \bar{\epsilon}_2 - \min\{d_1, d_2\})] \]
\[ \geq 0 \]

where the first inequality follows from the definition of \( X_1 \) and \((h_1, d_1) \in X_1 \), the second inequality follows from the statement that lottery \( A \) is preferred to lottery \( B \) (equivalently, that the payoff function has increasing differences in \([(h,d), \bar{\epsilon}] \) (using Remark 1)), the third inequality follows from the assumption that \( u \) and \( v \) are concave, so the payoff function is supermodular, and the fourth inequality follows from the definition of \( X_2 \) and \((h_2, d_2) \in X_2 \). The conclusion then follows. Furthermore, by Lemma 1 in the main body of the paper lottery \( A \) is preferred to lottery \( B \) for all \( h' \geq h'' \), \( d' \geq d'' \), and all \( \bar{\epsilon}_1 \succ \bar{\epsilon}_2 \) if and only if \( v^{(2)}(h'\epsilon - d') [h']^2 - v^{(2)}(h''\epsilon - d'') [h'']^2 \leq 0 \) for all \( \epsilon \) (i.e. \( v^{(2)}(h\epsilon - d)h^2 \) is decreasing in \((h,d)) \).\[ \square \]

**Example 6. Risky R&D investments**

Define \( X_i = \arg\max_{x \in B} EU(x, \bar{\epsilon}_i) = \arg\max_{(q,r) \in B} E[p(q) - c(r\bar{\epsilon}_i) q - g(r)] \), \( i = 1,2 \), and \( \bar{\epsilon}_1 \succ \bar{\epsilon}_2 \). Suppose that \( B \) is a compact sublattice of \( \mathbb{R}^2 \) and that \( c(\epsilon r) \) is twice differentiable. If \( c^{(1)}(\epsilon r) < 0 \) and \( c^{(2)}(\epsilon r) qr^2 \) is increasing in \((q,r) \) (e.g. if \( c^{(2)}(\epsilon r) \) is positive and non-decreasing), then, the profit function is supermodular in \((q,r) \) and has increasing differences in \([(q,r), \bar{\epsilon}] \); thus, \( X_1 \geq_S X_2 \).
Proof. The fact that supermodularity of the profit function in \((q,r)\) and increasing differences in \([(q,r), \tilde{c}]\) implies \(X_1 \geq S X_2\) can be shown along the same lines of Example 5 (or directly from Proposition 3). Thus, we only prove the statement above. First note that the profit function is supermodular in \((q,r)\) if and only if the function \(-c(r\tilde{c}_i)q\) is supermodular in \((q,r)\). Thus, the profit function is supermodular if marginal cost is decreasing in \(r\). Similarly, by Remark 1 in the main body of the paper, it suffices to consider the condition under which the function \(-c(r\tilde{c}_i)q\) displays increasing differences in \([(q,r), \tilde{c}]\) : \([-c(r'\tilde{c}_1)q' - c(r''\tilde{c}_2)q''] \geq [-c(r'\tilde{c}_2)q' - c(r''\tilde{c}_1)q'']\), or equivalently, \([c(r'\tilde{c}_2)q' + c(r''\tilde{c}_1)q''] \geq [c(r'\tilde{c}_1)q' + c(r''\tilde{c}_2)q'']\) for all \(q' \geq q'', r' \geq r''\), and all \(\tilde{c}_1 \geq \tilde{c}_2\). By Lemma 1 in the main body of the paper this will be the case if \(c^{(2)}(r'\tilde{c})q' [r']^2 - c^{(2)}(r''\tilde{c})q'' [r'']^2 \geq 0\) (i.e. \(c^{(2)}(r\tilde{c})q r^2\) is increasing in \((q,r)\)).