This paper presents a gift-exchange labor market model in which workers are prone to exhaustion. Although norms of reciprocity and fairness indicate that firms and workers must exchange ‘gifts’ in the form of higher wages and higher effort respectively, these incentives are mitigated by the evolution of fatigue and, therefore, the dynamic efficiency wage is lower than its static counterpart.

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Keywords: Fatigue; Efficiency wages

JEL classification: J24; J30

1. Introduction

Both classical and early neoclassical economists recognized that fatigue may not only decrease the marginal product of an additional hour of labor but also, since it accumulates over time, the long-run average product of a labor-day may decrease as the hours of work increase.1 This simple fact, which was vindicated by an intense stream of academic research in the early twentieth century (see, e.g., Florence, 1917; Vernon, 1921), has been for the most part ignored in formal mathematical treatments of the labor market.2

The objective of this note is to reconsider the issue of industrial fatigue. We provide a framework of labor market decisions with two main features. First, we consider a gift-exchange market, as in Akerlof (1982, 1984). Workers behave in a way that complies with the social norm, providing effort in excess of the minimum required – the workers’ ‘gift’ in reciprocity to the employer’s ‘gift’ in the form of higher wages.3 Second, workers are prone to exhaustion. Specifically, we characterize fatigue as limited mental (or physical) energy, a renewable resource that is utilized by the workers in the process of production.

We demonstrate that the dynamic efficiency wage that incorporates the evolution of fatigue is lower than its static counterpart. Intuitively, fatigue reduces efficiency and mitigates the incentive of the firm to reciprocate the workers’ ‘extra payment’ on top of those that received ‘no gifts’ tended to diminish after the first few hours of work and eventually efficiency was almost undistinguishable between the two treatments.4
2. Efficiency wages with mental energy as a factor of production

2.1. Workers

The behavior of the workers is a stylized version of the gift-exchange model presented by Akerlof (1982), to which the reader is referred for an in depth discussion of its motivation. Workers derive utility from consumption (the wage rate), their effort, and the effort norm \( e_n \). If they are to be employed they have to provide a minimum effort \( e_{\min} \), the ‘work rules’, but they are otherwise unconstrained in the amount of effort that they can exert. Workers ‘delegate’ the problem of mental energy allocation across time to the firm and therefore their problem is simply to work if

\[
\text{Max } \mu(w, e, e_n) \geq \mu(w),
\]

where \( w \) is the unemployment benefit.

In particular, suppose that the representative worker derives disutility from deviating from the effort norm such that the utility function takes the form, \( \mu(w, e, e_n) = w - (e - e_n)^2 \). In this case it is almost trivial that, as long as \( w > w_e \), the worker selects work and exert a level of effort \( e = e_n \). It remains to formulate how the effort norm is determined. Following Akerlof (1982), the effort norm is \( e_n = e_n(w, \phi) \), with \( \partial e_n / \partial w > 0 \) and \( \phi \) is a parameter that incorporates other determinants of the norm beyond the wage rate (e.g., the wage paid by other firms, unemployment benefits, and the unemployment rate). Intuitively, the norm says that if the firm provides a gift in the form of higher wages workers must demonstrate reciprocity by exerting effort above the minimum required.

In summary, effort takes the form \( e = e(w, \phi) \). In the next subsection we will use this effort supply function to solve the firm’s problem.

2.2. Employers

During a brief space of time, if a worker exerts more effort he or she will produce a larger amount of output. The main significance of fatigue is that it generates an intertemporal tradeoff between the current and future levels of output.

To formalize such tradeoff, suppose that the instantaneous level of production takes the form

\[
F(e(w), m, N), \tag{1}
\]

where \( e(w) \) is the effort supply function as described above [we suppressed the dependence on \( \phi \) for notational clarity], \( m \) is the stock of mental energy, and \( N \) the number of workers. The partial derivatives satisfy \( F_e > 0, F_v < 0 \), \( F_{en} > 0 \), \( e_{en} < 0 \), \( F_m > 0 \), \( F_{mm} < 0 \), \( F_{NN} > 0 \), \( F_{NN} < 0 \), so the production function has decreasing returns to labor and to wages and constant or decreasing returns in the stock of energy.

The stock of mental energy follows the process

\[
\dot{m} = v(m) - G(e(w), m, \theta). \tag{2}
\]

\( v(m) \) is the instantaneous change in mental energy in the absence of effort, and it satisfies \( v(0) = v(K) = 0 \), where \( K \) is a physiological constraint in the maximum carrying capacity of energy. The workers’ level of fatigue is defined as the difference between their current stock of energy and \( K \) and we label \( v(m) \) the rate of fatigue recovery. The rate of utilization of mental energy \( G(.) \) depends on the effort exerted, with \( G_m(.) > 0 \), on the stock of mental energy, with \( G_m(.) > 0 \), and possibly on other exogenous factors \( \theta \) (e.g. the intrinsic load of the work done).

The firm’s problem, letting the price of output equal one, is

\[
\text{Max } \int_0^\infty [F(e(w), m, N) - wN\exp^{-\psi}dt], \tag{3}
\]

subject to Eq. (2) and \( m(0) = m_0 \).

The current value Hamiltonian for the problem is

\[
H = F(e(w), m, N) - wN + \psi[v(m) - G(e(w), m)]. \tag{4}
\]

The first two terms in the RHS of Eq. (4) represent the static profit maximization problem. By increasing the wage rate the firm is able to increase the amount of effort exerted by the worker, increasing the current level of output. The third term shows, however, that such higher effort increases the rate of energy depletion, reducing the future productive capabilities of the workers.

The optimality conditions are

\[
F_e e_w = N + \psi G_e e_w, \tag{5}
\]

\[
F_N = w, \tag{6}
\]

\[
\dot{\psi} = \psi \phi - F_m - \psi(v_m - G_m), \tag{7}
\]

and the transversality condition: \( \psi > 0 \) and \( \exp^{-\psi} = \psi^{-0} \).

Eq. (5) asserts that the firm sets wages in a way that equalizes the marginal benefit of higher wages, i.e., higher effort exerted by the workers and therefore higher output, to the marginal costs, i.e., an increase in total costs and a higher utilization of mental energy. Eq. (6) says that the firm hires workers until the marginal product of an additional worker equals its marginal cost, the wage rate. Finally, Eq. (7) is the costate condition showing the evolution of the shadow price of the stock of mental energy.

2.3. Steady state equilibrium

In this note we concentrate the analysis on steady state relationships, in which case we have \( \dot{m} = 0 \) and \( \dot{\psi} = 0 \), or from Eqs. (2) and (7),

\[
v(m) = G(e(w), m, \theta), \tag{8}
\]

\[
\psi = \frac{F_m}{\phi + G_m - v_m}. \tag{9}
\]
Using Eqs. (5)–(9) we obtain the dynamic effort-wage elasticity \( e_{ew} \),

\[
e_{ew} = \frac{e_{FN}}{e_{Fe} - \rho e_{Ge}},
\]

with \( \rho = \frac{e_{gm}}{\phi_{(w)} + e_{gm} - e_{ew}} \),

where \( e_{FN}, e_{Fe}, \) and \( e_{gm} \) represent, respectively, the elasticity of output with respect to employment, to effort, and to mental energy, \( e_{Ge} \) and \( e_{Gm} \) are the elasticities of energy utilization with respect to effort and to the stock of energy, and \( e_{ew} \) is the elasticity of fatigue recovery with respect to the stock mental energy.

If mental/physical energy does not enter the production process, i.e. \( e_{Fm} = 0 \), we would obtain that the effort-wage elasticity equals \( e_{FN}/e_{Fe} \), as in Ramaswamy and Rowthorn (1991). They argue that, in general, \( e_{Fw} \) will be larger than \( e_{FN} \) due to workers’ complementarities, making the effort-wage elasticity lower than one and wages higher than in the celebrated Solow (1979) condition (\( e_{ew} = 1 \)). They label the difference with that condition the ‘damage potential of workers’ actions’.

The term \( \rho e_{Ge} \) incorporates the dynamic effect of wages. Since an increase in wages increases the utilization of mental energy it generates a disincentive to raise wages, decreasing the damage potential of workers’ actions. We can therefore conclude:

**Corollary.** The dynamic efficiency wage which incorporates the evolution of the workers’ energy resources is lower than its static counterpart.

The result follows from the concavity of the effort function.

2.4. Example

In this subsection we characterize the elements that may give rise to wage dispersion according to the parameters of a specific functional form of the workers’ energy renewal process. Suppose that employment is constant and normalized to one, that the production function takes the form \( F(e(w), m) = e(w)m \), and that the utilization of mental energy is proportional to output: \( G(.) = \gamma e(w)m \), where \( \gamma \) represents the working load (which may also include working conditions). Furthermore, we assume that the rate of recovery from fatigue obeys a logistic function \( v(m) = \delta m (1 - m/K) \), where \( \delta \) is the intrinsic growth rate of mental energy and \( K \) is the maximum carrying capacity. This functional form is widely used in the renewal resources literature; the fact that it constrains population growth in the absence of harvest makes it particularly suited to characterize the evolution of mental energy, which does not ‘reproduce’ infinitively in the absence of effort. In this case, the evolution of the stock is

\[
\dot{m} = m\delta (1 - m/K) - \gamma e(w)m. \tag{11}
\]

In steady state we have that \( \dot{m} = 0 \), which gives the ‘bioeconomic’ equilibrium level of mental energy as a function of effort \( m = K(1 - \gamma e(w)/\delta). \) It follows that output is

\[
F(e(w), m) = e(w, \phi)K(1 - \gamma e(w)/\delta). \tag{12}
\]

This steady state level of production incorporates the traditional contention (see Marshall’s introductory quotation) that, in the long-run, output increases over a certain range of values of effort-wages — i.e., when \( e \in [0, (\delta/2\gamma)] \), but if effort is sufficiently high output decreases and eventually becomes zero, i.e., when \( e \geq (\delta/\gamma) \).

Given Eqs. (11) and (12), the equilibrium wage is implicitly determined by

\[
e_{ew}(w)K \left( 1 - \frac{\gamma e(w)}{\delta} \right) \left( \frac{\varphi + \delta - 2\gamma e(w)}{\varphi + \delta - \gamma e(w)} \right) = 1. \tag{13}
\]

We have the following comparative statics of the equilibrium wage with respect to the parameters of the stock renewal process and time preference,

\[
\frac{\partial w}{\partial K} < 0, \quad \frac{\partial w}{\partial \delta} > 0, \quad \frac{\partial w}{\partial \gamma} < 0, \quad \frac{\partial w}{\partial \varphi} > 0.
\]

The intuition is simple, higher carrying capacity of mental energy, a faster intrinsic rate of recovery, and less intensive working loads reduce the dynamic loss in output due to fatigue and, therefore, wages increase. Also, if the firm is more impatient it will prefer a higher level of current output and therefore will select higher efficiency wages.

**Appendix A**

**Derivation of Eq. (13).**

The Hamiltonian is

\[
H = e(w, \phi)m - w + \psi (m\delta (1 - m/K) - \gamma e(w, \phi)m). \tag{A.1}
\]

The optimality conditions are

\[
e_{ew}(w, \phi)m(1 - \gamma \psi) = 1 \tag{A.2}
\]

\[
\dot{\psi} = \psi \varphi - e(w, \phi) - \psi (\delta (1 - 2m/K) - \gamma e(w, \phi)). \tag{A.3}
\]

In steady state we have \( \dot{\psi} = 0 \) and \( \dot{m} = 0 \), in which case we can write Eq. (A.3) as

\[
\psi = \frac{e(w, \phi)}{\varphi + \delta - \gamma e(w, \phi)}. \tag{A.4}
\]

Using Eq. (A.4) and the steady state value of \( m \) to substitute into Eq. (A.2) gives Eq. (13) in the text.

**References**