Precautionary Saving and Endogenous Labor Supply With and Without Intertemporal Expected Utility

Diego Nocetti  
Assistant Professor of Economics and Financial Studies  
School of Business, Clarkson University  
dnocetti@clarkson.edu

William T. Smith  
Professor of Economics  
Department of Economics, The University of Memphis  
wsmith@memphis.edu

Abstract

We analyze precautionary saving behavior in a framework with labor and non-labor income risks, an endogenous supply of labor, and a representation of preferences that disentangles attitudes towards risk, attitudes towards intertemporal substitution, and ordinal preferences for consumption and leisure. This preference structure allows us to disentangle and to describe in an intuitive way the different forces that determine precautionary saving “in the small” and “in the large”.

* JEL Classification: D91, E21, J22

* Keywords: Precautionary Saving, Wage Uncertainty; Labor Supply, Multivariate Risk Aversion, Inter-temporal Substitution

* We thank two anonymous referees for making excellent suggestions and comments that helped us to improve the manuscript.
When future income is uncertain, consumers may select to accumulate additional wealth as a precaution. Kimball (1990) gave the name prudence to the sensitivity of saving to risk and proposed the index of absolute prudence, $-v''/v'(x)/v''(x)$, and the index of relative prudence, $-v''/(x)v''(x)$, as measures of this sensitivity. Kimball and Weil (2009) generalized these measures by adopting a preference structure where risk and intertemporal substitution preferences are separated. Both Kimball (1990) and Kimball and Weil (2009) worked under the assumption of exogenous labor income. This assumption is quite restrictive. It implies, in particular, that the consumer can respond to uncertainty only by adjusting his current consumption. If the consumer can also adjust his supply of labor he may instead (or in addition) work more today. Alternatively, he may decide to work more in the future if faced with an adverse shock.

Evaluating the interaction between precautionary saving and labor supply is not a simple task. For example, the endogeneity of labor allows consumers to hedge movements in income. Given this self-insurance function of labor, one may expect individuals to be “less prudent”. Yet, greater saving allows the consumer to better self insure since at larger levels of income the consumer can respond more efficiently to changes in wages. Furthermore, by saving and or working more when young, workers reduce the need to work in the future and are able to reduce the variability of future labor income. It is unclear, a priori, if and under what conditions workers will save as a precaution for the uncertain future wages.

Second, if preferences are defined over both consumption and leisure, then – in a world with uncertainty -- the shape of the period utility function governs both risk preferences and ordinal preferences over the two goods. It becomes necessary to define multi-commodity measures of risk preferences. Furthermore, given the dynamic aspect of saving, it is necessary to use a preference structure that disentangles risk preferences from preferences towards intertemporal substitution, as in Kimball and Weil (2009).

In this paper we develop a model with three features:
• Labor supply is endogenous
• There is uncertainty surrounding both labor and non-labor income (e.g. exogenous medical expenses).
• Preferences are rich enough to disentangle ordinal preferences for consumption and leisure, attitudes towards risk, and attitudes towards intertemporal substitution.

To tackle the separation of risk aversion and intertemporal substitution we adopt, as Kimball and Weil (2009) did, the ordinal-certainty-equivalent (OCE) representation proposed by Selden (1978, 1979). To distinguish ordinal preferences from risk and intertemporal smoothing preferences we follow the classic literature on multivariate risk aversion developed by Kihlstrom and Mirman (1974, 1981).

These methods allow us to characterize the period preferences in a particularly simple form: period indirect-ordinal-utility is linear in full income. As a result, when non-labor income is uncertain, the general properties of the precautionary saving premium are the same as those derived by Kimball and Weil (2009). For example, decreasing absolute risk aversion (DARA), when properly defined in our multi-commodity model, is sufficient for prudence and it is also necessary for a precautionary saving premium larger than the risk premium. Relative to the case of certainty, consumers save and work more at a young age as a precaution, and they also work more when faced with an adverse income shock.

Because of the dual role of wages as income and as the relative price of leisure, the analysis is more involved when wages are uncertain. We show, however, that given the optimal levels of leisure and consumption, the problem is equivalent to analyzing precautionary saving with exogenous income and a random interest rate (that is correlated to income and that is higher under uncertainty). This decomposition is useful because it allows us to map well known results about saving with exogenous income risk [e.g. Leland (1968), Sandmo (1970), Drèze and Modigliani (1972), Kimball (1990), Kimball and Weil (2009)] and with interest rate risk [e.g. Rothschild and Stiglitz (1971), Selden (1979), Weil (1990), Eeckhoudt and Schlesinger (2008)] to the case of wage risk with endogenous labor supply.
We also show that, as in the case with exogenous income risk, the precautionary saving motive under wage uncertainty is closely related to the properties of the risk premium. In particular, the consumer is prudent if the risk premium is positive and decreasing. Unlike the case of exogenous income risk, risk aversion is not sufficient for a positive risk premium. If labor supply is sufficiently elastic, workers may gain from wage variability. Furthermore, under wage risk DARA is sufficient but not necessary for a decreasing risk premium. Therefore, the precautionary premium is larger than the risk premium under weaker conditions than with exogenous income risk. In other words, the substitution effect of Drèze and Modigliani (1972), which captures the incentive to save beyond what one would expect from the change in utility caused by uncertainty, is positive under weaker conditions. This occurs because greater saving allows consumers to reduce the need to work hard in adverse times and it improves the leisure-consumption trade-off.

Our paper is closely related to the work of Floden (2006). Floden presents small risk approximations of precautionary wealth in a model with wage uncertainty and intertemporal expected utility. Our analysis of precautionary saving with wage uncertainty is different from his in at least two important aspects. First, we present results about precautionary saving both “in the small” and “in the large”. Second, our preference structure allows us to disentangle and to describe in an intuitive way the different forces that determine precautionary saving. This separation is critical for understanding precautionary saving. For example, using a Cobb-Douglas specification of preferences, Floden showed that the degree of risk aversion (or the resistance to intertemporal substitution) has a U-shaped effect on precautionary wealth.¹ So, for example, the precautionary saving motive is highest for those individuals that are least averse to risk (and to intertemporal fluctuations). This unintuitive result disappears once we separate risk preferences from intertemporal smoothing preferences: an increase in risk aversion that also increases absolute prudence (e.g. under constant relative risk aversion), or a decrease in the resistance to intertemporal substitution, strengthens the precautionary saving motive.

¹ Low (2005) obtained the same result using a numerical analysis of a life cycle model.
The rest of the paper is organized as follows. In the Section 1 we present the model and we provide an in-depth discussion of our preference structure. In Section 2 we analyze precautionary saving with non-labor income uncertainty. In Section 3 we first evaluate the properties of the risk premium under wage uncertainty and then analyze precautionary saving with and without intertemporal expected utility. Section 4 provides numerical examples of the local measures of prudence while Section 5 concludes.

1. THE MODEL

1.1 Budget Constraints and Preferences

Our model generalizes the framework of Floden (2006) and of Kimball and Weil (2009). Our consumer lives two periods. He receives a non-stochastic wage \( w_1 \) in the first period, but faces both a stochastic wage \( w_2 \) and stochastic non-wage income \( y_2 \) in the second period. Non-wage income could be interpreted, for example, as exogenous medical expenditures or as government transfer payments. The prices of the consumption good in the two periods are constant; for simplicity we use consumption as numéraire, and set \( p_1 = p_2 = 1 \). The consumer selects consumption and leisure \( c_i, l_i, i = 1,2 \) in each period and savings \( x \). To simplify the algebra we assume that the rate of return on saving is zero and that the individual has an endowment of one unit of time in each period. Labor supply is thus \( n = 1 - l \).

The budget constraints for the two periods are

\[
\begin{align*}
    c_1 &= (1 - l_1)w_1 - x \\
    c_2 &= y_2 + (1 - l_2)w_2 + x
\end{align*}
\]

where \( w_2 = \bar{w}_2 + \tilde{w}, y_2 = \bar{y}_2 + \tilde{y} \) and \( \bar{w} \) and \( \bar{y} \) are mean zero random variables with variances \( \sigma_{\bar{w}}^2 \geq 0 \) and \( \sigma_{\bar{y}}^2 \geq 0 \) and covariance \( \sigma_{yw} \). The distributions of both income and the wage are bounded away from zero. For expositional simplicity, we assume \( w_1 = \bar{w}_2 = \bar{w}, \bar{y}_2 = 0, \) and \( \sigma_{yw} = 0. \)
We consider a specification of utility that allows us to disentangle the three aspects of preferences to which we have alluded: risk aversion, resistance to intertemporal substitution, and ordinal preferences over consumption and leisure. Our utility function is

$$u[f(c_1, l_1)] + U\{v^{-1}[Ev[f(c_2, l_2)]]\}.$$  

(3)

We assume that $f(\cdot)$ is linearly homogeneous, with positive and diminishing marginal utilities, $f_c > 0, f_l > 0, f_{cc} < 0, f_{ll} < 0$. For simplicity we also impose strict concavity, $f_{c}f_{ll} - f_{cl}^2 > 0$. The functions $u(\cdot), U(\cdot),$ and $v(\cdot)$ are monotonically increasing and concave: $u' > 0, u'' < 0, U' > 0, U'' < 0, v' > 0, v'' < 0.$

These preferences incorporate two key elements. First, to disentangle risk preferences from ordinal preferences over the two goods we apply the methods of Kihlstrom and Mirman (1974, 1981). Second, to disentangle risk preferences from intertemporal smoothing preferences we apply the preference specification of Selden (1978, 1979). We will explain each of these aspects in some depth.

1.1.1 Disentangling Risk Preferences from Ordinal Preferences

In their classic work on multivariate risk aversion Kihlstrom and Mirman (1974, 1981) addressed the fundamental question of what risk aversion means when there are multiple goods in the utility function. The problem is that the curvature of the utility function governs both risk aversion and preferences over the goods. How can these aspects of preference be disentangled? Kihlstrom and Mirman’s (1981) answer rested upon two pillars. First, Pratt (1964) showed that, if utility is defined over a single good, person A is more risk averse than person B if and only if A’s utility function is more concave than B’s. Second, Debreu (1976) showed that any convex representation of ordinal preferences that can be expressed as a concave utility function will possess a “least concave” representation. Kihlstrom and Mirman demonstrated that if the utility function is homothetic then this least concave representation is linearly homogeneous.

---

2 Since marginal utilities are homogeneous of degree zero, a simple application of Euler’s theorem reveals that diminishing marginal utility requires $f_{cl} > 0$. 

Notice that the function \( f(\cdot) \) in Eq. (5) is linearly homogeneous and that the function \( \nu(\cdot) \) is monotone increasing. It follows that \( V(c_2, l_2) = \nu[f(c_2, l_2)] \) is a homothetic function of \( c_2 \) and \( l_2 \).\(^3\) The linearly homogeneous function \( f(\cdot) \) is the “least concave” expression of ordinal preferences. The function \( \nu(\cdot) \) induces a concave transformation of the underlying \( f(\cdot) \) preferences and so – à la Pratt – makes them more risk averse. The upshot is that we can interpret \( f(\cdot) \) as governing ordinal preferences over the different goods and the transforming function \( \nu(\cdot) \) as governing atemporal risk preferences. Alternatively, we can think of the curvature of \( \nu(\cdot) \) as determining risk aversion with respect to the “aggregator” \( f(\cdot) \).

Since \( \nu(\cdot) \) governs risk aversion, we will refer to \( A = -\nu''/\nu' \) as absolute risk aversion and to \( R(f) = -\nu''/f/\nu' \) as relative risk aversion. Following Kimball and Weil (2009) we will call \( \varepsilon(f) = -A' f/A \) the elasticity of risk tolerance. We also follow their example in noting for future reference that \( R(f) \) and \( \varepsilon(f) \) are related by the identity

\[
R(f) + \varepsilon(f) = P(f),
\]

(4)

where \( P(f) = -\frac{\nu''}{\nu'} f \) is the measure of relative prudence under intertemporal expected utility maximization. We also define the function

\[
M = \nu^{-1}\left[Ev[f(c_2, l_2)]\right].
\]

(5)

This is certainty-equivalent ordinal utility in the second period. It will play an important role in what follows.

1.1.2 Disentangling Risk Preferences from Intertemporal Smoothing Preferences

In another classic pair of papers Selden (1978, 1979) suggested a class of preferences that allow atemporal risk preferences to be distinguished from intertemporal preferences over riskless consumption streams.\(^4\) In Selden’s model a consumer lives two periods and faces random

---

\(^3\) This is the definition of homotheticity (Simon and Blume, 1994).

\(^4\) Kreps and Porteus (1978) developed a closely related, multi-period model. As Kimball and Weil (2009) point out, the two-period version of Kreps-Porteus preferences is equivalent to that of Selden under very general assumptions.
consumption in the second period. The consumer has a standard, atemporal von-Neumann-Morgenstern expected utility function defined over second period consumption, \( E\nu(c_2) \). The consumer first calculates the certainty-equivalent of second period consumption, \( M = v^{-1}[E\nu(c_2)] \). He then maximizes lifetime utility defined over first-period consumption and the certainty-equivalent of second-period consumption:

\[
u(c_1) + U(M). \tag{6}\]

To relate this to our preferences, notice that Eq. (3) can be expressed as

\[
u[f(c_1, l_1)] + U(M). \tag{7}\]

Our consumer maximizes lifetime utility defined over first-period ordinal utility [the aggregator \( f(\cdot) \)] and certainty-equivalent second-period utility \( M \). Following Kimball and Weil (2009), we will refer to \( s(f) = -U'/fU'' \) as the elasticity of intertemporal substitution and to \( 1/s(f) \) as the degree of resistance to intertemporal substitution.

\subsection*{1.1.3 An Example}

A useful example is the power/Cobb-Douglas case, which we will use later:

\[
u[f(c_1, l_1)] + U\{v^{-1}[E\nu[f(c_2, l_2)]]\} = \frac{(c_1^{a-b}\ell_1^b)^{1-\gamma}}{1-\gamma} + \delta \frac{[E(c_2^{a-b}\ell_2^b)^{1-\gamma}]^{1-\rho}}{1-\rho}, \tag{8}\]

where \( \gamma > 0, \rho > 0 \), and \( \delta \) is the discount factor.\(^6\) The functions inside the parentheses are linearly homogeneous. It follows that the correct measure of relative risk aversion in this case is \( R = \gamma \); similarly, the correct measure of the elasticity of intertemporal substitution is \( s = 1/\rho \).

\textbf{Remark.} In a different framework, Smith (2001) uses the more general Cobb-Douglas specification \( f(c_1, l_1) = c_1^a l_1^b \) with \( a + b \leq 1 \). He shows that the “effective” coefficient of

\(^5\) We use Kimball and Weil’s convention of having a second period utility over the certainty equivalent consumption different from the first period utility over consumption to allow for time discounting.

\(^6\) As usual, \( v = \ln f \) when \( \gamma = 1 \) and \( u = \ln f \) when \( \rho = 1 \).
relative risk aversion in this case is \( R = 1 - (a + b)(1 - \gamma) \). Similarly, the “effective” elasticity of intertemporal substitution is \( s = 1/[1 - (a + b)(1 - \rho)] \). Unless otherwise specified, we will restrict \( a + b = 1 \) as in Eq. (8) for notational simplicity.

1.2. The Consumer’s Problem

The consumer chooses \( c_1, l_1, c_2, l_2 \), and \( x \) in order to maximize Equation (3) given the budget constraints (1) and (2). We emphasize that \( c_2 \) and \( l_2 \) are chosen after the realization of \( \tilde{w} \) and \( \tilde{y} \). To solve this problem, it is convenient to decompose it, as Floden (2006) does, into two parts.

**Step 1**: Maximize utility in each period *given* savings.

In the first period the consumer maximizes \( u[f(c_1, l_1)] \) subject to Equation (1). Since \( u(\cdot) \) is monotone we can generate the same demand functions by solving

\[
\max_{c_1, l_1} f(c_1, l_1) \ s.t. \ c_1 + w_1 l_1 = z_1 = w_1 - x,
\]

(9)

where \( z_1 \) is Beckerian “full” income. It is well known that if the utility function is homothetic, the resulting Marshallian demand functions will be proportional to income, \( c_1^*(w_1, z_1) = g_c(w_1) z_1, \ l_1^*(w_1, z_1) = g_l(w_1) z_1 \). Furthermore, the indirect utility function is also proportional to income, \( \bar{f}(w_1, z_1) = h(w_1) z_1 \).

Similarly, the consumer maximizes utility in the second period subject to the budget constraint (2), *given* savings \( x \), and *after* the realization of uncertainty. This means the second-period decision reduces to the simple, static, non-stochastic problem

\[
\max_{c_2, l_2} f(c_2, l_2) \ s.t. \ c_2 + w_2 l_2 = z_2 = w_2 + y_2 + x,
\]

(10)

where, to reiterate, \( w_2 \) and \( y_2 \) are already known in the second period. This yields the Marshallian demands \( c_2^*(w_2, z_2) = g_c(w_2) z_2, \ l_2^*(w_2, z_2) = g_l(w_2) z_2 \) and the indirect utility function \( \bar{f}(w_2, z_2) = h(w_2) z_2 \).

\[\text{7 In the absence of uncertainty it is trivially the case that } v^{-1}Ev(f) = f. \text{ That the second order condition holds follows immediately from the concavity of } f(\cdot).\]
Step 2: Choose savings to maximize lifetime utility.

Substituting the indirect utility functions from the two static problems into the lifetime utility function yields

$$\max_x u[h(w_1)(w_1 - x)] + U[v^{-1}[E v[h(w_2)(w_2 + y_2 + x)]]]. \quad (11)$$

Note that certainty-equivalent ordinal utility can now be expressed as a function of $x$ alone. We therefore write

$$M(x) = v^{-1}[E v[h(w_2)(w_2 + y_2 + x)]]. \quad (12)$$

This permits the consumer’s problem to be expressed more succinctly as

$$\max_x u[h(w_1)(w_1 - x)] + U[M(x)]. \quad (13)$$

The first-order condition for this problem is

$$u'[h(w_1)(w_1 - x)]h(w_1) = U'[M(x)]M'(x). \quad (14)$$

where $M'(x) = \frac{Eh(w_2)v'[h(w_2)(w_2 + y_2 + x)]}{v[M(x)]}$. As for the second-order condition, it is well known that the objective function in models of precautionary saving without intertemporal expected utility maximization may not be well-behaved [Gollier (2001, p. 301), Kimball and Weil (2009)]. However, a sufficient condition for lifetime utility to be concave with respect to $x$ is that certainty-equivalent ordinal utility be concave with respect to $x$. We will assume this condition is satisfied.

To analyze how uncertainty affects saving and labor supply behavior imagine a consumer confronted with uncertainty about either wages or exogenous income. Given the concavity of $M(x)$, the first-order condition (Eq. 14) implies that risk of either sort will increase saving if the marginal benefit of saving is greater under uncertainty than under certainty:

$$U'[M(x)]M'(x) > U'[h(\tilde{w})(\tilde{w} + x)]h(\tilde{w}). \quad (15)$$
To determine the strength of the precautionary saving motive we follow Kimball and Weil (2009) in defining the (compensating) precautionary premium $\theta^*$ by the extra amount of certain income that would induce the consumer to save the same amount, $x^*$, with uncertainty as if there were no uncertainty:

$$U / [M(x^* + \theta^*)] M / (x^* + \theta^*) = U / [h(\bar{\omega})(\bar{\omega} + x^*)] h(\bar{\omega}).$$ \hfill (16)

In the following sections we study the local and global properties of $\theta^*$ and we compare it with the compensating precautionary premium $\psi^*$ that holds under intertemporal expected utility, which is defined as [Kimball (1990)]

$$Ev / [h(w_2)(w_2 + y_2 + \psi^* + x^*)] h(w_2) = v / [h(\bar{\omega})(\bar{\omega} + x^*)] h(\bar{\omega}).$$ \hfill (17)

For future reference we also introduce the compensating risk premium $\pi^*$, which satisfies

$$Ev[h(w_2)(w_2 + y_2 + \pi^* + x)] = v[h(\bar{\omega})(\bar{\omega} + x)].$$ \hfill (18)

2. PURE INCOME UNCERTAINTY

We first consider “pure” income uncertainty, where non-wage income is random but the wage is nonstochastic ($\sigma_y > 0, \sigma_w = 0$). This will both highlight how our preference structure with multiple commodities alters the interpretation of the results in Kimball and Weil (2009) and provide a benchmark against which to compare the effects of wage uncertainty ($\sigma_y = 0, \sigma_w > 0$), which we consider in the Section 3.

In the absence of wage uncertainty, the wage is equal to $\bar{\omega}$, its expected value in the presence of uncertainty. In the ensuing discussion it will be important to remember that this causes certainty equivalent ordinal utility to be a function of both the saving rate and the expected wage:

$$M(x, \bar{\omega}) = v^{-1}[Ev[h(\bar{\omega})(\bar{\omega} + y_2 + x)]] \hfill (19)$$

This in turn means that the first-order condition becomes
\[ U'[M(x, \bar{w})]M'(x, \bar{w}) = u'[h(\bar{w})(\bar{w} - x)]h(\bar{w}), \quad (20) \]

where \( M'(x, \bar{w}) = \frac{E u'[h(\bar{w})(\bar{w} + \hat{\gamma} + x)]}{u'[M(x, \bar{w})]'} h(\bar{w}). \)

Given a risk compensated by the precautionary premium \( \theta^* \) as defined in (16), the left hand side of (20) equals the marginal utility of saving under certainty. In Appendix A.1 we show that a local approximation of the precautionary premium is

\[ \theta^* = \left\{ A[h(\bar{w})\bar{z}_2] + \frac{U'[h(\bar{w})\bar{z}_2]}{U'[h(\bar{w})\bar{z}_2]} A'[h(\bar{w})\bar{z}_2] \right\} h(\bar{w}) \frac{\sigma^2_y}{2} + o\left(\sigma^2_y\right), \quad (21) \]

where \( o\left(\sigma^2_y\right) \) collects terms that go to zero faster than \( \sigma^2_y \). Using the definitions of the elasticity of intertemporal substitution and the elasticity of risk tolerance, and suppressing the dependence on the expected wage to avoid notational clutter, this becomes

\[ \theta^* = Ah\{1 + s\varepsilon\} \frac{\sigma^2_y}{2} + o\left(\sigma^2_y\right), \quad (21)' \]

or equivalently,

\[ \theta^* = R\{1 + s\varepsilon\} \frac{\sigma^2_y}{2\hat{z}_2} + o\left(\sigma^2_y\right). \quad (21)'' \]

This is similar to the local compensating premium in Kimball and Weil [2009, Equation (14)]. As in their paper, the precautionary saving motive depends upon attitudes towards risk, \( R(\cdot) \) and \( \varepsilon(\cdot) \), and attitudes towards intertemporal substitution \( s(\cdot) \). In particular, it is still true that decreasing absolute risk aversion, which implies \( \varepsilon \geq 0 \), is sufficient for a positive precautionary premium.

DARA is also necessary for the local precautionary premium to be larger than the local risk premium, \( \pi^* = Ah \frac{\sigma^2_y}{2} \). Drèze and Modigliani (1972) were the first to prove this result in the context of exogenous income uncertainty and intertemporal expected utility, while Kimball and Weil (2009) showed that the result still holds with Selden preferences. As Kimball and Weil
(2009 pg.253) point out, “decreasing absolute risk aversion means that greater saving makes it more desirable to take on a compensated risk. But the other side of such a complementarity between saving and a compensated risk is that a compensated risk makes saving more attractive.” Drèze and Modigliani (1972) referred to the incentive to save beyond what one would expect from the reduction of utility as a substitution effect. Following Kimball (1990) we will refer to this effect as the “Drèze-Modigliani substitution effect.” It is captured in the local precautionary premium by the term \( \frac{R \sigma_e^2}{\bar{z}_2} \).

Furthermore, the local precautionary premium with Selden preferences is larger than the precautionary premium under intertemporal expected utility if

\[
R \{1 + s \epsilon\} \geq P = R + \epsilon,
\]

a condition that holds if absolute risk aversion is decreasing and \( R \geq 1/s \), i.e. relative risk aversion is larger than the resistance to intertemporal substitution, or if absolute risk aversion is increasing and \( R \leq 1/s \). Kimball and Weil (2009) further show that these conditions are valid globally. The following proposition establishes that this is also true in our framework.

**Proposition 1 (Kimball and Weil).** Given exogenous income risk, DARA implies precautionary saving.\(^8\) The precautionary premium \( \theta^* \) is larger than the risk premium \( \pi^* \) if and only if preferences exhibit DARA, and it is larger than \( \psi^* \) if preferences exhibit DARA and \( R \geq 1/s \) or if absolute risk aversion is increasing and \( R \leq 1/s \). Under constant absolute risk aversion, \( \theta^* = \pi^* = \psi^* \).

**Proof.** See Appendix A.2.

Since the first and second period demands for leisure are decreasing and increasing, respectively, in the saving rate Proposition 1 has the following corollary,

\[^8\text{Another local and global condition for precautionary saving is } v'/// \succeq 0 \text{ and relative risk aversion smaller than the resistance to intertemporal substitution, } R \leq 1/s \text{ [Gollier (2001, pg 302), Kimball and Weil (2009)].}\]
**Corollary.** *If individuals exhibit DARA, an increase in non-labor income risk increases the current supply of labor and decreases the expected future supply of labor.*

As an example, suppose that we adopt the power/Cobb-Douglas utility function as in Eq. 8. In this case relative risk aversion is $R = \gamma$ and the elasticity of intertemporal substitution is $s = 1/\rho$. Since this function has constant relative risk aversion, which implies $\varepsilon = 1$, we obtain

$$
\theta^* = \gamma [1 + 1/\rho] \frac{\sigma^2}{2\hat{s}_2}.
$$

(22)

Eq. (22) is identical to the local precautionary premium obtained by Kimball and Weil (2009) using a single-attribute utility function.

Despite the similarities between Kimball and Weil’s results and ours, one should not conclude that the precautionary saving motive is generally the same when leisure enters the utility function as when it does not. Equation (21) warns us that it is only the transforming function $v(\cdot)$ that determines preferences towards risk. This should not be confounded with ordinal preferences between consumption and leisure. Similarly, it is the transforming function $U(\cdot)$ that governs the elasticity of intertemporal substitution $s$. In other words, preferences for risk and intertemporal substitution still affect the premium as Kimball and Weil (2009) predict they will, *provided* that we look at the curvature properties of the right functions.

Consider, for instance, the more general power/Cobb-Douglas function with the aggregator $f(c_t, l_t) = c^a_t l^b_t$ and $a + b \leq 1$. As argued above, the effective degree of relative risk aversion in this case is $R = 1 - (a + b)(1 - \gamma)$ and the effective elasticity of intertemporal substitution is $s = 1/[1 - (a + b)(1 - \rho)]$. Under this specification we obtain

$$
\theta^* = [1 - (a + b)(1 - \gamma)] [1 + 1/(1 - (a + b)(1 - \rho))] \frac{\sigma^2}{2\hat{s}_2}.
$$

(23)

Clearly, if $a + b < 1$ the local precautionary premium with multiple commodities is different from the premium with a single commodity.
Alternatively, consider the preference specification adopted by Weil (1993), where absolute risk aversion is constant, \( v[f] = -\frac{e^{-\alpha f}}{a} \), and the elasticity of intertemporal substitution is also constant, \( U[M] = \frac{(M)(\rho)}{1-\rho} \). Furthermore, suppose that non-labor income is normally distributed with mean zero and variance \( \sigma_y^2 \). Under these assumptions there is an explicit solution for the precautionary saving premium,

\[
\theta^* = \alpha h(\bar{w}) \frac{\sigma_y^2}{2}. 
\]  

(24)

As implied by Proposition 1, when risk preferences exhibit constant absolute risk aversion, intertemporal smoothing preferences do not affect precautionary saving, but ordinal preferences still do. Therefore, the strength of the precautionary saving motive is in general different from that when leisure does not enter the utility function. For example, Equation (24) implies that the precautionary saving motive will be low when labor income is high. In a framework where consumers derive utility only from consumption and income is exogenous, we would instead conclude that the precautionary saving motive is independent of expected income.

3. WAGE UNCERTAINTY

In this section we evaluate precautionary saving and labor supply when second period wages are stochastic and non-wage income is certain. Recall that the first-order condition is

\[
u[f(h(w_1), w_1 - x)]h(w_1) = U'[M(x)]M'(x),
\]

where

\[
M'(x) = \frac{E_h(w_2)\psi[h(w_2)(w_2 + x)]}{\psi[M(x)]}. 
\]

The effect of wage uncertainty on the marginal utility of saving reflects the dual role of wages as income and as the relative price of leisure, captured by the term \( h(w_2) \). As explained by Eaton (1980), an increase in relative price variability is equivalent to an increase in the riskiness of the

\[\text{For example, if the aggregator } f(.) \text{ is Cobb-Douglas with } a + b = 1 \text{ this expression becomes,}\]

\[
\theta^* = a(1 - b)^{1-b} b^b \bar{w}^{-b} \frac{\sigma_y^2}{2}.
\]
rate of return on savings and, due to the quasi-convexity of indirect utility in prices, \( h_{ww}(\cdot) \geq 0 \),
to an increase in the expected return on savings. Following Eaton (1980), we will refer to these
two aspects as the “capital risk” effect and the “expected return” effect. We will refer to the fact
that relative prices are perfectly correlated with income as the “covariance effect”. This
decomposition is useful because it will allow us to map some well known results about saving
with exogenous income risk and with interest rate risk (and our results in Section 2) to the case
of wage risk with endogenous labor supply.

We can alternatively analyze the precautionary saving motive under wage uncertainty by
evaluating the properties of the risk premium. Recall that with income uncertainty DARA is
sufficient for prudence and it is also necessary for a precautionary saving motive stronger than
risk aversion (with and without intertemporal expected utility). We could re-phrase this to say
that if the risk premium is decreasing the consumer is prudent and the precautionary premium is
larger than the risk premium. That is, the Drèze-Modigliani substitution effect is positive if the
risk premium, more generally than absolute risk aversion, is decreasing. We will show that
essentially the same intuition holds when the source of risk is labor income and labor supply is
endogenous. To do so we first evaluate the effect of wage uncertainty on expected utility. Then,
we analyze precautionary saving under intertemporal expected utility and under the more general
Selden preferences.

### 3.1 Wage Uncertainty and Welfare

When labor supply is exogenous, risk averse consumers dislike mean preserving increases in
income risk. When labor supply is endogenous, however, this may not be the case. In particular,
wage uncertainty decreases second-period expected utility if \( v[h(w)z_2] \) is a concave function of
\( w \). Differentiating twice this function and suppressing the dependency on wages and saving, we
obtain the following condition

\[
\frac{v'}{[h_wz_2 + h]^2} + \frac{v' h_{ww} z_2 + 2h_w}{[h_{ww} z_2 + 2h_w]} \leq 0.
\]  

(25)

To interpret this expression we will first rewrite it as follows
\[
\frac{v''}{v'} h z_2 \left[ \frac{h w z_2}{h} + 1 \right]^2 + \frac{h w}{k} z_2 \left[ \frac{h w w z_2}{h w} + 2 \right] \leq 0. \tag{25'}
\]

Now notice that, by Roy’s identity, the second-period Marshallian demand for leisure given a wage rate \( w \) is \( l_2 = -\frac{h w z_2}{h} \). We will also define \(-\frac{h w w}{h w} w \equiv \kappa > 0\). The function \( \kappa \) measures the curvature of indirect ordinal utility with respect to the relative price of leisure. It is simple to show that closer substitutability or a higher wage elasticity of leisure increase the level of \( \kappa \).\(^{10}\)

Using these definitions we obtain the following necessary and sufficient condition for wage uncertainty to decrease expected utility

\[
R(1 - l_2)^2 + l_2 \left( 2 - \frac{\kappa z_2}{w} \right) \geq 0. \tag{26}
\]

We can therefore conclude

**Proposition 2.** Wage risk decreases expected utility if and only if \( R(1 - l_2)^2 + 2l_2 \geq \kappa l_2 \frac{z_2}{w} \).

**Proof.** See above.

The ambiguous effect of wage uncertainty on expected utility arises from the well known result [e.g. Hanoch (1977), Eaton (1980), Turnovsky et al.,(1980)] that relative price variability may increase consumer’s welfare. Intuitively, the consumer can self-insure against negative price shocks by adjusting the demand of the commodity. Inequality (26) shows that, in our framework where prices are correlated with income, welfare is more likely to increase if the degree of risk aversion is low, labor supply is low under certainty, and consumption and leisure are close substitutes (so \( \kappa \) is high).

Although this result has some interest on its own, it will be particularly useful to establish conditions for wage uncertainty to induce precautionary saving. For later reference, we note that a local approximation of the risk premium around \( w_2 = \bar{w} \) is

\(^{10}\) Specifically, \( \kappa \) is related to the wage elasticity of leisure, \( \varepsilon_{l,w} = -l_w w / l_2 \), by the identity \( \kappa = \varepsilon_{l,w} + \frac{w}{z_2} (1 + l_2) \), and to the elasticity of intra-temporal substitution between consumption and leisure, \( s_{ct} \), by the identity \( \kappa = 1 + l_2 \frac{w}{z_2} - \left( 1 - l_2 \frac{w}{z_2} \right) (1 - s_{ct}) \).
\[ \pi^* = R(1 - l_2)^2 + l_2 \left( 2 - \frac{\kappa \bar{z}_2}{\bar{w}} \right) \frac{\sigma_w^2}{2\bar{z}_2}. \]  

(27)

Comparing (27) with the local risk premium under exogenous income uncertainty, \( \pi^* = Ah \frac{\sigma^2}{2} \), we observe that the counterpart to \( Ah \) when labor income is stochastic is

\[ \hat{A} \equiv Ah(1 - l_2)^2 + \frac{l_2}{\bar{z}_2} \left( 2 - \frac{\kappa \bar{z}_2}{\bar{w}} \right). \]

Importantly, \( \hat{A} \) depends on both risk preferences and ordinal preferences. To reiterate, it may be negative.

**Remark.** If the aggregator \( f(\cdot) \) is Cobb-Douglas and \( x = 0 \) under certainty we have \( l_2 = b \), \( \kappa = 1 + b \), and \( \bar{z}_2 = \bar{w} \). Therefore, we obtain

\[ \hat{A}_{c-D} = \left[ R(1 - b)^2 + b(1 - b) \right] \frac{1}{\bar{w}}, \]

which is unambiguously positive. In this case the degree of substitutability between consumption and leisure is relatively low (\( \kappa \) is sufficiently low), so wage risk decreases welfare.

### 3.2 Intertemporal Expected Utility

We defined the compensating precautionary premium under intertemporal expected utility in Equation (17). To evaluate the precautionary premium we will first consider the case in which the aggregator \( f(\cdot) \) is Cobb-Douglas and \( x = 0 \) under certainty.

#### 3.2.1 Cobb-Douglas Utility

When the aggregator \( f(\cdot) \) is Cobb-Douglas and \( x = 0 \) under certainty we can rewrite Equation (17) as

\[ Ev'[Bw_2^{-b}(w_2 + \psi_{c-D}^*)]w_2^{-b} = v'[B\bar{w}^{-b}]\bar{w}^{-b}, \]

(28)

with \( B = (1 - b)^{1-b}b^b \).
A Taylor series approximation of Eq. (28) around $w_2 = \bar{w}$ and $\psi_{C-D}^* = 0$ gives

$$\{v///[.]\bar{w}^{-3b}B^2(1-b)^2 - 3v///[.]B\bar{w}^{-1-2b}b(1-b) + v///[.]\bar{w}^{-2-b}b(1+b)\} \frac{\sigma^2_w}{2} = -v///[.]B\bar{w}^{-2b}\psi_{C-D}^*. \tag{29}$$

Using the fact that Cobb-Douglas preferences imply that $P = -\frac{v///}{v/}B\bar{w}^{-1-b}$ and $R = -\frac{v/}{v/}B\bar{w}^{-1-b}$ and solving for $\psi_{C-D}^*$ we obtain

$$\psi_{C-D}^* = \{P(1-b)^2 + 3b(1-b) + b(1+b)/R\} \frac{\sigma^2_w}{2\bar{w}^2}. \tag{30}$$

Therefore, for the Cobb-Douglass case, the local counterpart to relative prudence when the wage rate is uncertain and the consumer maximizes intertemporal expected utility is

$$\varphi_{C-D}^I \equiv P(1-b)^2 + 3b(1-b) + b(1+b)/R. \tag{31}$$

Under large risks, $\varphi_{C-D}^I > 0$ is also necessary and sufficient for precautionary saving since the marginal utility of saving, $v/[Bw^{-b}(w+x)]w^{-b}$, is convex in $w$ if and only if $\varphi_{C-D}^I > 0$ when $x = 0$. The following proposition summarizes this result.

**Proposition 3.** Suppose that the consumer maximizes intertemporal expected utility, that ordinal preferences are Cobb-Douglas, and that $x = 0$ under certainty. Then, wage uncertainty induces precautionary saving if and only if $\varphi_{C-D}^I > 0$, as defined in Eq. (31), is positive.

**Proof.** See above.

Proposition 3 has two important implications. First, the condition for precautionary saving with wage uncertainty is weaker than the equivalent condition with exogenous income risk. In particular, for risk averse consumers $v/// > 0 \iff P > 0$ is sufficient, but not necessary, for a positive precautionary saving motive under wage uncertainty. We will see later that this is true more generally for those utility functions for which wage uncertainty decreases expected utility.
Second, if relative prudence is increasing in the degree of risk aversion, e.g. under constant relative risk aversion in which case \( P = 1 + R \), the effect of risk aversion on the precautionary premium is U-shaped. That is, the precautionary saving motive is highest when risk aversion is either very low or very high. Low (2005) and Floden (2006) also obtained this result. This is quite unintuitive. One would expect that, at least locally and when relative risk aversion is constant, more risk averse agents would save more. We will show later that this unintuitive result disappears when we disentangle risk aversion from intertemporal smoothing preferences.

3.2.2 General Case

Consider now the more general case. It is simple to show that, if the risk is small, the precautionary premium can be written as

\[
\psi^* = \left\{ P + (l_2)^2(P - 2) - \kappa l_2 \left( 1 - \frac{1}{R} \frac{\sigma^2}{w} - 2l_2(P - 2) \right) \right\} \frac{\sigma^2}{w^2}.
\]  

(32)

Therefore, the local counterpart to relative prudence when the wage rate is uncertain is

\[
\phi^t \equiv P + (l_2)^2(P - 2) - \kappa l_2 \left( 1 - \frac{1}{R} \frac{\sigma^2}{w} - 2l_2(P - 2) \right).
\]  

(33)

The different terms in (32) and (33) correspond to the decomposition of wage uncertainty mentioned at the beginning of Section 3. Specifically,

- The term \( P \) represents the effect of income uncertainty while maintaining the relative price of leisure, and so the rate of return on saving, constant. As usual given an exogenous income risk [Leland (1968), Sandmo (1970), Kimball (1990)], it is positive for risk averse consumers if \( \psi^t > 0 \).

- \( (l_2)^2(P - 2) \) is the capital risk effect. It is positive if relative prudence is larger than two [Rothschild and Stiglitz (1971), Eeckhoudt and Schlesinger (2008)], or, under constant relative risk aversion, if relative risk aversion is larger than one.
\[ -\kappa l_2 \left( 1 - \frac{1}{R} \right) \frac{z^2}{w} \] reflects the expected return effect. Wage uncertainty increases the expected rate of return on savings, which increases optimal saving if relative risk aversion is smaller than one [e.g. Gollier 2001, Proposition 63].

\[ -2l_2 (P - 2) \] is the covariance effect, which is positive if relative prudence is smaller than 2. This effect reflects the fact that a negative shock to income is partially hedged by a decrease in the relative price of leisure, which reduces the incentive to save for sufficiently prudent consumers.

**Remark.** Empirical estimates of relative risk aversion are typically larger than two. Given the empirically plausible case of DARA, relative prudence is larger than risk aversion, so the most likely case is that \( P > 2 \). This implies that the income risk effect and the capital risk effect are positive and that the expected return effect and the covariance effect are negative.

Combining terms in (33) we obtain,

\[
g^0 = P(1 - l_2)^2 + 2l_2(2 - l_2) - \kappa l_2 \left( 1 - \frac{1}{R} \right) \frac{z^2}{w}. \tag{33}' \]

Eq. (33)' implies that a random interest rate combined with a random (and perfectly correlated) exogenous income will induce precautionary saving under intertemporal expected utility (given \( P \geq 0 \)). Unlike the Cobb-Douglas case, if the elasticity \( \kappa \) is sufficiently high (consumption and leisure are close substitutes, so the wage elasticity of leisure is high), the precautionary premium can be negative -provided \( R > 1 \).

The following proposition establishes the equivalent result for large risks.

**Proposition 4.** Under intertemporal expected utility wage uncertainty induces precautionary saving if and only if

\[
P(1 - l_2)^2 + 2l_2(2 - l_2) - \kappa l_2 \left( 1 - \frac{1}{R} \right) \frac{z^2}{w} > 0.\]
**Proof.** \( v' [h(w)(w + x)] h(w) \) is convex in \( w \) if and only if the condition in the proposition holds.

An obvious corollary of the proposition is that \( P \geq 0 \) and \( R \leq 1 \) are sufficient for precautionary saving to occur. The problem with this condition is that \( R \leq 1 \) is not very likely empirically. We will now present an alternative sufficient condition by evaluating the properties of the risk premium.

Given a small risk we can rewrite the precautionary premium as

\[
\psi^* = \varrho \sigma_{\tilde{w}}^2 = \bar{A} \left( 1 + \frac{1}{R} \right) \frac{\sigma_{\tilde{w}}^2}{2},
\]

(34)

where we defined \( \bar{\varepsilon} = -\left( \partial \bar{A} / \partial x \right) (\bar{z}_2 / \bar{A}) \). \( \bar{\varepsilon} \) measures the income elasticity of the risk premium. This alternative specification allows us to decompose the precautionary premium into two components: (1) The risk premium, \( \bar{A} \sigma_{\tilde{w}}^2 \), and (2) The Drèze-Modigliani substitution effect, \( \bar{A} \left( \frac{1}{R} \right) \frac{\sigma_{\tilde{w}}^2}{2} \).

Clearly, the Drèze-Modigliani substitution effect is positive, and the local precautionary premium is larger than the local risk premium, if the risk premium is decreasing, \( \left( \partial \bar{A} / \partial x \right) < 0 \). Now notice that the local risk premium is decreasing if

\[
- \left( \partial \bar{A} / \partial x \right) \bar{z}_2 = \varepsilon A h (1 - l_2)^2 + 2 Ah (1 - l_2) l_2 + \kappa (l_2 / \tilde{w}) \geq 0
\]

(35)

Condition (35) is weaker than decreasing absolute risk aversion, \( \varepsilon > 0 \), the necessary and sufficient condition for \( \psi^* > \pi^* \) when the source of risk is exogenous income. For example, if absolute risk aversion is constant the precautionary premium is still larger than the risk premium. Intuitively, when labor income is stochastic and labor supply is endogenous, in addition to decreasing absolute risk aversion, there are two reasons for greater saving to make a compensated risk more desirable. First, an increase in the saving rate increases second-period leisure, endogenously reducing the degree of uncertainty. Second, an increase in the saving rate
increases the price elasticity of leisure, allowing the consumer to substitute consumption and leisure more efficiently. In other words, greater saving allows consumers with flexibility to reduce the need to work hard in adverse times and it improves the leisure-consumption trade-off.

The same is true under large risks. In particular,

**Proposition 5.** Under intertemporal expected utility and wage uncertainty, if the risk premium is decreasing (e.g. under non-increasing absolute risk aversion), the precautionary premium is larger than the risk premium.

**Proof.** See Appendix B.2.

If, in addition, the risk premium is positive we have $\psi^*(x) \geq \pi^*(x) \geq 0$. We can then conclude

**Proposition 6.** Assume that the consumer maximizes intertemporal expected utility and that the risk premium is positive (Proposition 2) and decreasing (Proposition 5). Then, wage uncertainty induces precautionary saving, increases the current supply of labor, and it decreases the expected future supply of labor.

**Proof.** See above.

As we observed before, when the aggregator $f(.)$ is Cobb-Douglas the risk premium is unambiguously positive, so DARA is sufficient for a positive precautionary saving premium in this case.

The potential difficulty in the interpretation of these results is that the effect of wage uncertainty clearly depends on both risk preferences and intertemporal smoothing preferences. To understand precautionary saving under wage uncertainty and endogenous labor supply we need a framework that disentangles those different aspects of preferences.
3.3 Selden Preferences

The intuition of the different forces that determine precautionary saving under wage risk and Selden preferences can be more easily explained if we start by assuming that the size of the risk is small. As before, we will start by evaluating the Cobb-Douglas case.

3.3.1 Cobb-Douglas Utility

When the aggregator $f(.)$ is Cobb-Douglas and $x = 0$ under certainty a local approximation of the precautionary premium is (see Appendix B.1)

$$\theta^*_c = R (1 + s \varepsilon) (1 - b)^2 + b (1 - b) (1 + 2 s R) + b (1 + b) s \frac{\alpha^2}{2 \omega} + o(\sigma^2_w).$$  \hspace{1cm} (41)

Just like the case with intertemporal expected utility, when ordinal preferences are Cobb-Douglas the precautionary premium under wage uncertainty is positive under weaker conditions than its counterpart with exogenous income uncertainty. Specifically, with Selden preferences $s \varepsilon \geq -1$ is sufficient, but clearly not necessary, for precautionary saving under wage uncertainty. The intuition is the same as in the case with intertemporal expected utility: By saving more consumers with labor flexibility reduce the degree of uncertainty and can substitute consumption and leisure more efficiently.

Eq. (41) also implies that an increase in risk aversion that increases both $R$ and $\varepsilon$ (so it increases $P$) or a decrease in the resistance to intertemporal substitution (an increase in $s$) will strengthen the precautionary saving motive. This is in sharp contrast to the intertemporal expected utility framework [e.g. Low (2005) and Floden (2006)]. As we saw above [Eq. (30)], restricting $R = 1/s$ implies that the effect of risk aversion on the precautionary premium is U-shaped. One could then conclude that savings are highest when individuals are least averse to risk (most resistant to intertemporal substitution), which is the opposite of what we found. Disentangling risk aversion from intertemporal substitution preferences is therefore critical for understanding precautionary saving under wage uncertainty.
3.3.2 General Case

More generally, the local approximation of the precautionary premium with Selden preferences is given by (see Appendix B.1)

\[
\theta^* = \left\{ R[1 + s\varepsilon] + R(l_2)^2[1 + s(\varepsilon - 2)] - \kappa l_2[1 - s] \frac{\bar{z}_2}{\bar{w}} \right\} \sigma_{w}^{2} + o(\sigma_{w}^{2}),
\]

(42)

and the local counterpart to relative prudence under the more general Selden preferences and wage risk is

\[
\theta_{s}^{*} = \left\{ R[1 + s\varepsilon] + R(l_2)^2[1 + s(\varepsilon - 2)] - \kappa l_2[1 - s] \frac{\bar{z}_2}{\bar{w}} \right\} \sigma_{w}^{2} + o(\sigma_{w}^{2}),
\]

(43)

The local measure of prudence under wage uncertainty and Selden preferences is a complicated expression determined by attitudes towards risk, attitudes towards intertemporal substitution, and preferences between attributes in the absence of uncertainty. Each of its terms, however, have an intuitive interpretation:

- \( R[1 + s\varepsilon] \) represents the effect of income uncertainty while holding the relative price of leisure constant. It is the same that we obtained above given Selden preferences and a pure income risk [Eq. (21)]. As argued there, decreasing absolute risk aversion is sufficient for this term to be positive.

- \( R(l_2)^2[1 + s(\varepsilon - 2)] \) captures the capital risk effect. It is positive if relative risk aversion is non-increasing, which implies \( \varepsilon \geq 1 \), and \( s < 1 \). As shown by Selden (1979) and by Weil (1990), under constant relative risk aversion capital risk increases saving only if \( s < 1 \). The degree of risk aversion only affects the magnitude of savings.

- \( -\kappa l_2[1 - s] \frac{\bar{z}_2}{\bar{w}} \) represents the expected return effect. It is positive only if \( s > 1 \); risk aversion plays no role whatsoever.
• $2l_2[1 + R(s(1 - \varepsilon) - 1)]$ accounts for the correlation between income and the relative price of leisure. Under constant relative risk aversion it is positive only if the degree of risk aversion is smaller than one. More generally, it also depends on attitudes towards intertemporal substitution and on the elasticity of risk tolerance.

**Remark.** The consensus is that $s$ is well below one [e.g. Hall (1988), Epstein and Zin (1991), Ogaki and Reinhart (1998)]. If, in addition, we assume that relative risk aversion is constant, the capital risk effect and the income risk effect are positive and the expected return effect and the covariance effect (given the more plausible case that $R > 1$) are negative.

As we did in the case with intertemporal expected utility, we can combine terms to show that a random interest rate combined with a random (and perfectly correlated) exogenous income will induce precautionary saving and that only the expected return effect may reduce the local precautionary premium,

$$\varrho^S = R(1 + s\varepsilon)(1 - l_2)^2 + 2l_2 \left(1 + Rs \left(1 - l_2\right)\right) - kl_2(1 - s)\frac{z_2}{w}.$$

(43)’

Alternatively, using the small risk approximation of the risk premium, we have

$$\varrho^S = \tilde{h}z_2(1 + s\hat{\varepsilon}).$$

(43)’’

From these two equivalent expressions we can observe that either $s > 1$ or $\tilde{h} > 0$ and $s\hat{\varepsilon} > 0$ are sufficient for a positive local precautionary premium. The following proposition establishes the equivalent result for large risks.

**Proposition 7.** With Selden preferences and wage uncertainty the precautionary premium is positive

1) *if absolute risk aversion is non-increasing (which is sufficient for a positive Drèze-Modigliani substitution effect) and the risk premium is positive, or,*
2) independent of the sign of the risk premium, if absolute risk aversion is non-increasing and the elasticity of intertemporal substitution is no smaller than one.\footnote{Another sufficient condition for precautionary saving under wage uncertainty and Selden preferences is that the precautionary premium with intertemporal expected utility is positive (which does not require DARA) and \(Rs \leq 1\). This is equivalent to the condition \(v''/v' h^2 \geq 0\) and \(Rs \leq 1\) that holds under income risk (see footnote 8).}

**Proof.** See Appendix B.3.

Given the current consensus that \(s\) is below one the second condition is unlikely to hold in practice. However, the consensus is also that labor supply is fairly inelastic (as in the Cobb-Douglas case), which is sufficient for the risk premium to be positive. Therefore, given the empirically plausible case of non-increasing absolute risk aversion, we should expect that, for most individuals, wage uncertainty will induce precautionary saving and will increase the labor supply of young workers. This is consistent with the findings of Parker, Barmby, and Belghitar (2005).

The next proposition, which is apparent from our representation of the local precautionary premium in terms of the risk premium and its income elasticity (the Drèze-Modigliani substitution effect) and which we prove in Appendix B.4 for the case of large risks, compares the strength of the precautionary saving motive with and without intertemporal expected utility,

**Proposition 8.** Suppose that \(\psi^* > \pi^*\), as it will be the case if absolute risk aversion is non-increasing (Proposition 5). Then, \(\theta^* \geq \psi^*\) according with \(R \geq 1/s\).

**Proof.** See Appendix B.4.

Therefore, given DARA, a utility function where risk preferences are intermingled with intertemporal smoothing preferences will overstate or understate the precautionary saving motive depending on whether risk aversion is weaker or stronger than the resistance to intertemporal substitution.
3.3.3 A Closed-Form Solution

It is well known that dynamic models with stochastic labor income and endogenous labor supply cannot be solved, in general, in closed form. With Selden preferences, however, it is possible to derive a closed form solution under the assumptions of risk neutrality and constant elasticity of intertemporal substitution. Specifically, under these assumptions one can solve explicitly for the precautionary premium,

\[ \theta^* = \frac{[h(\overline{w})]^{1-s} \bar{z}_2 - Eh(w_2) \bar{z}_2 [Eh(w_2)]^{-s}}{[Eh(w_2)]^{1-s}}. \]  

(44)

Consistent with Proposition 7, if wage uncertainty decreases expected utility, \( Eh(w_2) \bar{z}_2 \leq h(\overline{w}) \bar{z}_2 \) (e.g. Cobb-Douglas utility and \( x^* = 0 \)), or, independent of ordinal preferences, if the elasticity of intertemporal substitution is no smaller than 1, the precautionary premium is positive. If neither of these conditions is satisfied, saving could be lower under wage uncertainty.\(^{12} \) \(^{13} \)

4. NUMERICAL EXAMPLES

It is instructive to illustrate some of our local results numerically. We assume that preferences are power/Cobb-Douglas as in Eq. (8), that \( \sigma^2_w = \sigma^2_y \), and evaluate the local precautionary premium under non-labor income [Eq. (22)] and the local precautionary premium under wage risk [Eq. (41)] for different values of \( \gamma \) and \( \rho \). Table 1 presents the precautionary premia under five different scenarios.

The first line in the table presents our preferred parameter specification, with an elasticity of intertemporal substitution equal to 0.05 (so \( \rho = 20 \)), a number close to Hall’s (1988) estimates,

\(^{12} \) This is also consistent with Proposition 8. Under risk neutrality \( Rs = 0 \) and the precautionary premium under intertemporal expected utility goes to infinity as risk aversion tends to zero, so \( \theta^* < \psi^* \).

\(^{13} \) We can go one step further if we assume that the aggregator \( f(\cdot) \) is Cobb-Douglas and that wages are log-normally distributed, \( \ln(w_2) \sim N \left( \ln \bar{w} - \frac{\sigma^2_w}{2}, \sigma^2_w \right) \). We also assume \( x^* = 0 \). In such case the precautionary premium is given by \( \theta^* = \frac{1 - \exp \left( - \frac{2b - b(1+b)(1-s)\sigma^2_w}{2} \right)}{\exp \left[ b(1+b)(1-s)\frac{\sigma^2_w}{2} \right] \bar{w}} \), which is unambiguously positive.

---

This is also consistent with Proposition 8. Under risk neutrality \( Rs = 0 \) and the precautionary premium under intertemporal expected utility goes to infinity as risk aversion tends to zero, so \( \theta^* < \psi^* \).

We can go one step further if we assume that the aggregator \( f(\cdot) \) is Cobb-Douglas and that wages are log-normally distributed, \( \ln(w_2) \sim N \left( \ln \bar{w} - \frac{\sigma^2_w}{2}, \sigma^2_w \right) \). We also assume \( x^* = 0 \). In such case the precautionary premium is given by \( \theta^* = \frac{1 - \exp \left( - \frac{2b - b(1+b)(1-s)\sigma^2_w}{2} \right)}{\exp \left[ b(1+b)(1-s)\frac{\sigma^2_w}{2} \right] \bar{w}} \), which is unambiguously positive.
and relative risk aversion equal to 3. In an intertemporal expected utility framework, restricting \( \gamma = \rho = 20 \) or \( \gamma = \rho = 3 \), the precautionary premium could be grossly overstated (Propositions 1 and 8). With wage uncertainty and a low degree of risk aversion the problem is particularly acute since, restricting the resistance to intertemporal substitution to also be low, implies that the precautionary premium is high (e.g. the precautionary premium is higher when \( \gamma = \rho = 0.5 \) than the case when \( \gamma = \rho = 3 \)). The same is not true in the case of non-labor income risk since, in this case, the local precautionary premium increases monotonically with the degree of relative risk aversion whether with or without intertemporal expected utility.

More generally, the precautionary saving motive under non-labor income risk is stronger than its counterpart for wage risk when relative risk aversion and the resistance to intertemporal substitution are both high (as it seems to be the case empirically) but it is weaker when they are both low. Intuitively, when both relative risk aversion and the resistance to intertemporal substitution are low, the magnitude of the expected return effect that arises under wage risk is positive and large relative to the other effects.

5. CONCLUSIONS

We analyzed precautionary saving in a model with labor and non-labor income uncertainty, endogenous labor supply, and a preference structure that disentangles ordinal preferences for consumption and leisure, attitudes towards risk, and attitudes towards intertemporal substitution. Under plausible assumptions, uncertainty of either sort induces precautionary saving, increases the current supply of labor, and it decreases the expected future supply of labor. We showed that the distinction among the different aspects of preferences is central for understanding how uncertainty affects precautionary saving, and it is also important for understanding how wage uncertainty differs from non-labor income uncertainty.

Our analysis provides the theoretical skeleton for evaluating other issues as well. For example, our framework could be used to analyze the effect of labor flexibility on the precautionary saving motive, in the spirit of Floden (2006). As we argued above, labor flexibility
provides insurance, which reduces the need to save as a precaution, but it is also complementary
to saving when there is wage uncertainty, in the sense that greater saving allows consumers with
flexibility to reduce the need to work hard in adverse times and it improves the leisure-
consumption trade-off. Floden (2006) shows that, under plausible specifications of preferences
and (small) wage uncertainty, individuals with flexibility save more than individuals with a fixed labor
schedule and the same preferences. In this case, the complementarity between precautionary saving and
labor flexibility is stronger than the self-insurance provided by a flexible labor schedule. In an appendix,
available upon request, we show that disentangling risk preferences from intertemporal smoothing
preferences does not change this important message.\textsuperscript{14} We also show, however, that with non-wage
uncertainty individuals with flexibility save less (given the empirical plausible case that the elasticity of
intertemporal substitution is low).\textsuperscript{15} It is therefore essential to distinguish the source of risk to understand
the effect of labor flexibility on precautionary saving. As Gollier (2005) points out, evaluating flexibility
under large risks is also critical for understanding behavior towards risk.

Our framework also lends itself to any problem involving dynamic choice under uncertainty
with multiple goods. For example, we suggested in the paper that a source of exogenous income
risk could be exogenous medical expenditures. The empirical literature that looks for a link
between health uncertainty and saving (e.g. Gruber and Yelowitz, 1999; Palumbo, 1999; Chou et
al. 2003) works under this assumption. If health is an argument in the utility function and
medical expenditures are endogenous, however, precautionary saving behavior may be very
different from the case of exogenous expenditures (especially if the risk is endogenous too, as in
the case of uncertain health care quality).

A second example is problems where environmental quality is an argument in the utility
function. For instance, Gollier (2010) analyzes the properties of the “ecological discount rate”,
the rate that should be utilized to discount environmental impacts. Using a utility function of the

\textsuperscript{14} We also show that the distinction is still important since, for any given level of the elasticity of intertemporal
substitution the premium under flexibility may be lower or higher than the premium under a fixed schedule if the
comparison is made among individuals with different degrees of risk aversion. Similarly, for any given level of risk
aversion the premium under a fixed schedule may be lower or higher if attitudes towards intertemporal smoothing
are different.

\textsuperscript{15} This is consistent with the findings of Bodie et al (1992), who showed that individuals with a flexible labor
schedule will select to invest a larger share of their wealth in the risky asset.
form $U(\text{consumption, environmental quality})$, he evaluates how shifts in the distributions of environmental quality or the consumption endowment affect the discount rate of economies with different preferences. He shows, for example, that in an economy where the consumption endowment is riskier, the ecological discount rate will be lower if there is “cross prudence”, as defined by Eeckhoudt et al’s (2007) ($U_{211} > 0$). Such interpretation, however, is clouded by the fact that the shape of the utility function governs risk preferences, intertemporal substitution preferences, and ordinal preferences. The methods in this paper could be used to disentangle how these different aspects of preferences interact to determine the ecological discount rate.

**APPENDIX A.**

**A.1. Derivation of the Local Precautionary Premium for “Pure” Income Uncertainty**

We extend the method employed by Kimball and Weil (2009). Following their example, we begin by deriving the “equivalent” premium, and then use it to derive the “compensating” premium in Equation (21) of the text.

To simplify notation denote mean “full” income in the second period by $\bar{z}_2 = \bar{w}_2 + \bar{y}_2 + x$. The “equivalent” premium $\theta(\bar{z}_2)$ is then defined by

$$U^\prime[M(x, \bar{w}_2)]M_x(x, \bar{w}_2) = U^\prime[h(\bar{w}_2)(\bar{z}_2 - \theta(\bar{z}_2))]h(\bar{w}_2)$$  \hspace{1cm} (A.1)

Now note that, for small income risks $\bar{y}$, certainty-equivalent ordinal utility can be expressed with the Taylor-series

$$M[h(\bar{w}_2)(\bar{z}_2 + \bar{y})] = h(\bar{w}_2)\left\{\bar{z}_2 - A[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2)\frac{\sigma^2}{2}\right\} + o(\sigma^2),$$  \hspace{1cm} (A.2)

where $A[h(\bar{w}_2)\bar{z}_2] = -\nu^\prime/[h(\bar{w}_2)\bar{z}_2]/\nu^\prime [h(\bar{w}_2)\bar{z}_2]$ is absolute risk aversion. Differentiating Equation (A.2) in turn implies that

$$M_x(s, \bar{w}_2) = h(\bar{w}_2)\left\{1 - A^\prime[h(\bar{w}_2)\bar{z}_2][h(\bar{w}_2)]^2\frac{\sigma^2}{2}\right\} + o(\sigma^2).$$  \hspace{1cm} (A.3)
Now substitute Equations (A.2) and (A.3) into Equation (A.1). Take Taylor series of, respectively, the left-hand side around \( M = h(\bar{w}_2)\bar{z}_2 \) (which holds in the absence of uncertainty) and of the right hand side around \( \theta = 0 \):

\[
\begin{align*}
&\left\{ U'[h(\bar{w}_2)\bar{z}_2] - U''[h(\bar{w}_2)\bar{z}_2][h(\bar{w}_2)]^2 A[h(\bar{w}_2)\bar{z}_2] \frac{\sigma_y^2}{2} \\
&\quad + o(\sigma^2)\right\}\left\{ 1 - A'/[h(\bar{w}_2)\bar{z}_2][h(\bar{w}_2)]^2 \frac{\sigma_y^2}{2} \right\} + o(\sigma^2) \right\}
&= U'[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2) - U''[h(\bar{w}_2)\bar{z}_2][h(\bar{w}_2)]^2 \theta + o(\sigma^2).
\end{align*}
\]

(A.4)

Simplifying Equation (A.4) and eliminating lower-order terms leads to

\[
\theta(s, \bar{w}_2) = \left[ A[h(\bar{w}_2)\bar{z}_2] + \frac{U'[h(\bar{w}_2)\bar{z}_2]}{U''[h(\bar{w}_2)\bar{z}_2]} A'[h(\bar{w}_2)\bar{z}_2] \right] h(\bar{w}_2) \frac{\sigma_y^2}{2} + o(\sigma^2).\quad (A.5)
\]

Comparing Equation (16) in the text with Equation (A.1) reveals the compensating and equivalent premia are related by

\[
\theta^*(x, \bar{w}_2) = \theta[x + \theta^*(x, \bar{w}_2), \bar{w}_2]\quad (A.6)
\]

Using Equation (A.5) this implies

\[
\theta^*(x, \bar{w}_2) = \left\{ A[h(\bar{w}_2)(\bar{z}_2 + \theta^*(x, \bar{w}_2))] + \frac{U'[h(\bar{w}_2)(\bar{z}_2 + \theta^*(x, \bar{w}_2))]}{U''[h(\bar{w}_2)(\bar{z}_2 + \theta^*(x, \bar{w}_2))]} A'[h(\bar{w}_2)(\bar{z}_2 + \theta^*(x, \bar{w}_2))] \right\} h(\bar{w}_2) \frac{\sigma_y^2}{2} + o(\sigma^2).\quad (A.7)
\]

Assuming that \( A' \) and \( U'/U'' \) are continuous around \( \bar{z}_2 \) this reduces to

\[
\theta^*(x, \bar{w}_2) = \left\{ A[h(\bar{w}_2)\bar{z}_2] + \frac{U'[h(\bar{w}_2)\bar{z}_2]}{U''[h(\bar{w}_2)\bar{z}_2]} A'[h(\bar{w}_2)\bar{z}_2] \right\} h(\bar{w}_2) \frac{\sigma_y^2}{2} + o(\sigma^2).\quad (A.8)
\]

Recalling the definitions of the intertemporal elasticity of substitution and the elasticity of risk tolerance, Equation (A.8) reduces to

\[
\theta^*(x, \bar{w}_2) = A[h(\bar{w}_2)\bar{z}_2][1 + s[h(\bar{w}_2)\bar{z}_2]e[h(\bar{w}_2)\bar{z}_2]]h(\bar{w}_2) \frac{\sigma_y^2}{2} + o(\sigma^2).\quad (A.9)
\]
**A.2. Proof of Proposition 1**

We apply the same reasoning as Kimball and Weil (2009, Section 3.1). The risk premium \( \pi^*(x) \) is defined by

\[
E v[h(\bar{w}_2)(\bar{z}_2 + \pi^*(x) + \bar{y})] = v[h(\bar{w}_2)\bar{z}_2].
\]  
(A.10)

It is identically true that

\[
M[x + \pi^*(x), \bar{w}_2] \equiv h(\bar{w}_2)\bar{z}_2. 
\]  
(A.11)

Differentiate this identity to find

\[
M/[x + \pi^*(x), \bar{w}_2] = \frac{h(\bar{w}_2)}{1 + \frac{d\pi^*(x)}{dx}} 
\]  
(A.12)

It is well known [for example, Gollier (2001, Proposition 4) that, given an exogenous income risk, \( \text{DARA} \Leftrightarrow \frac{d\pi^*(x)}{dx} \leq 0 \). It follows that \( M/[x + \pi^*(x), \bar{w}_2] \geq h(\bar{w}_2) \). Now we can infer

\[
U/[M[x + \pi^*(x), \bar{w}_2]]M/[x + \pi^*(x), \bar{w}_2] = U/[h(\bar{w}_2)\bar{z}_2]M/[x + \pi^*(x), \bar{w}_2] \geq \\
U/[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2). 
\]  
(A.13)

However, since the marginal benefit of saving is assumed to be decreasing and \( \pi^*(x) \geq 0 \) under risk aversion

\[
U/[M(x, \bar{w}_2)]M/(x, \bar{w}_2) > U/[M[x + \pi^*(x), \bar{w}_2]]M/[x + \pi^*(x), \bar{w}_2]. 
\]  
(A.14)

Therefore

\[
U/[M(x, \bar{w}_2)]M/(x, \bar{w}_2) \geq U/[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2), 
\]  
(A.15)

which is inequality (15) in the text, the condition for precautionary saving to occur.

Now notice also that
\[ U'[M[x + \pi^*(x), \bar{w}_2]] M'[x + \pi^*(x), \bar{w}_2] \geq U'[h(\bar{w}_2)\bar{z}_2] h(\bar{w}_2) = U'[M[x + \theta^*(x), \bar{w}_2]] M'[x + \theta^*(x), \bar{w}_2], \] (A.16)

which implies, given a decreasing marginal utility of saving, \( \theta^* \geq \pi^* \).

To prove the last part of the proposition we also follow Kimball and Weil (2009). Write

\[ U'[M(x, \bar{w}_2)] M'[x, \bar{w}_2] \equiv \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + \bar{y} + x)] h(\bar{w}_2) = \]
\[ \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] h(\bar{w}_2). \] (A.17)

where \( \Lambda = u[v^{-1}(\cdot)] \) Then, using the precautionary premium under intertemporal expected utility \( \psi^* \) we have,

\[ \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] h(\bar{w}_2) = \]
\[ \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + x)] h(\bar{w}_2)]. \] (A.18)

Furthermore, if \( \Lambda \) is concave and \( \psi^* \geq \pi^* \), which holds under DARA, we have

\[ \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + \psi^* + \bar{y} + x)] h(\bar{w}_2) \leq \Lambda'[v[h(\bar{w}_2)(\bar{w}_2 + x)] h(\bar{w}_2)]. \] (A.19)

Since by definition,

\[ \Lambda'[E v[h(\bar{w}_2)(\bar{w}_2 + \theta^* + \bar{y} + x)] E v'[h(\bar{w}_2)(\bar{w}_2 + \theta^* + \bar{y} + x)] h(\bar{w}_2) = \]
\[ v'[h(\bar{w}_2)(\bar{w}_2 + x)] h(\bar{w}_2), \] (A.20)

we must have \( \psi^* \geq \theta^* \).

If \( \Lambda \) is convex and \( \psi^* \leq \pi^* \), which holds under increasing absolute risk aversion, we have the same inequalities. If, on the other hand, \( \Lambda \) is concave and \( \psi^* \leq \pi^* \) or if \( \Lambda \) is convex and \( \psi^* \geq \pi^* \) the inequalities would be reversed. Finally, note that concavity (convexity) of \( \Lambda \) is equivalent to \( R \leq (\geq) 1/s \).
APPENDIX B

B.1 Derivation of the Local Precautionary Premium for Wage Uncertainty

The equivalent premium is defined by

\[
U'/M(x)M_x(x) = U'/h(\bar{w}_2)(\bar{z}_2 - \theta^*(\bar{z}_2))h(\bar{w}_2).
\]  

(B.1)

Certainty-equivalent ordinal utility can be expressed with the Taylor-series

\[
M(x) = h\bar{z}_2 - (A(h + h'\bar{z}_2)^2 - h''\bar{z}_2 - 2h')\frac{\sigma_\omega^2}{2},
\]  

(B.2)

Differentiating this with respect to \(x\) we obtain

\[
M_x(x) = h - \left\{A'/h(h + h'\bar{z}_2)^2 + 2Ah'(h + h'\bar{z}_2) - h''\right\}\frac{\sigma_\omega^2}{2}
\]  

(B.3)

Now substitute Equations (B.2) and (B.3) into Equation (B.1). Take Taylor series of, respectively, the left-hand side around \(M = h(\bar{w})\bar{z}_2\) (which holds in the absence of uncertainty) and of the right hand side around \(\theta = 0\):

\[
\left[U/[, - U'[, A(h + h'\bar{z}_2)^2 - h''\bar{z}_2 - 2h')\frac{\sigma_\omega^2}{2} + o(\sigma^2)]h - \left\{A'h(h + h'\bar{z}_2)^2 + 2Ah'(h + h'\bar{z}_2) - h''\right\}\frac{\sigma_\omega^2}{2} + o(\sigma^2) = U/[, h - U'[, h^2\theta + o(\sigma^2)
\]  

(B.4)

Simplifying and using the definition of the compensating premium gives

\[
\theta^* = \left\{Ah(1 + s\epsilon)\left(\frac{h'h'\bar{z}_2}{h}\right)^2 - \frac{h''\bar{z}_2 + 2h'}{h} - 2As\frac{h'h'\bar{z}_2}{h} + \frac{h''\bar{z}_2}{h}\right\}\frac{\sigma_\omega^2}{2} + o(\sigma_\omega^2)
\]  

(B.5)

Roy’s identity implies that \(l_2 = -\frac{h'\bar{z}_2}{h}\). Define \(\kappa = -\frac{h_{ww}}{h_w}\bar{w}\). Therefore, we can write the premium as
\[
\theta^* = \left\{ Ah(1 + s \varepsilon)(1 - l_2)^2 + 2 l_2 + 2 R s l_2 (1 - l_2) - \kappa l_2 \frac{\bar{w}_2}{\bar{w}} (1 - s) - \frac{\sigma^2}{2 \bar{w}^2} \right\} + o(\sigma^2),
\]

(B.6)

which is Equation 42 in the text. The Cobb-Douglas case with \( x = 0 \) is obtained by setting \( l_2 = b, \bar{z}_2 = \bar{w} \), and simplifying.

**B.2. Proof of Proposition 5**

We will first show that DARA implies \( \frac{d\pi^*(x)}{dx} \leq 0 \) and then that \( \frac{d\pi^*(x)}{dx} \leq 0 \iff \psi^*(x) \geq \pi^*(x) \).

Recall that the compensating risk premium is defined by

\[
M[x + \pi^*(x)] = h(\bar{w}_2) \bar{z}_2.
\]

Differentiate this identity to find

\[
\frac{M'[x+\pi^*(x)]}{h(\bar{w}_2)} = \frac{E v'[h(w_2)(w_2 + x + \pi^*(x))]h(w_2)}{v'[M[x+\pi^*(x)]h(\bar{w}_2)]} = \frac{1}{1 + \frac{d\pi^*(x)}{dx}}.
\]

(B.7)

Notice first that

\[
\text{cov}[v'[h(w_2)(w_2 + x + \pi^*(x))], h(w_2)] \geq 0
\]

since \( h(w_2) \) decreases with wages, \( h(w_2)(w_2 + x + \pi^*(x)) \) increases with wages as long as \( l \leq 1 \), and \( v \) is concave. We also know that \( E h(w_2) \geq h(\bar{w}_2) \) given the quasi-convexity of indirect utility in prices. It follows that

\[
E v'[h(w_2)(w_2 + x + \pi^*(x))]h(w_2) \geq E v'[h(w_2)(w_2 + x + \pi^*(x))]h(\bar{w}_2).
\]

(B.8)

Furthermore, if absolute risk aversion is non-increasing we have

\[
\frac{E v'[h(w_2)(w_2 + x + \pi^*(x))]}{v'[M[x+\pi^*(x)]]} \geq 1,
\]

(B.9)

which implies, together with Eq. (B.8), that \( \frac{d\pi^*(x)}{dx} \leq 0 \).

We can now infer that

\[
E v'[h(w_2)(w_2 + x + \pi^*(x))]h(w_2)
\]

We can now infer that.
\[ \geq v'/M[x + \pi^*(x)]h(\bar{w}_2) \]
\[ = v'/(h(\bar{w}_2)\bar{z}_2)h(\bar{w}_2) \]
\[ = E v'/[h(w_2)(w_2 + x + \psi^*(x))]h(w_2), \quad (B.10) \]

where the first equality follows from the definition of the risk premium and the second equality follows from the definition of the precautionary premium under intertemporal expected utility (see Eq. 17). We can conclude that
\[ E v'/[h(w_2)(w_2 + x + \pi^*(x))]h(w_2) \geq E v'/[h(w_2)(w_2 + x + \psi^*(x))]h(w_2). \quad (B.11) \]

Since the function \( v(.) \) is concave we must have \( \psi^*(x) \geq \pi^*(x) \).

### B.3. Proof of Proposition 7

The first part of the proposition says that if absolute risk aversion is non-increasing and the risk premium is positive, wage uncertainty induces precautionary saving. To prove this recall that the necessary and sufficient condition for precautionary saving is
\[ U/M[x]M'[x] \geq U/[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2). \]
The risk premium being positive implies that \( M[x] \leq h(\bar{w}_2)\bar{z}_2 \). We established the condition for a positive risk premium in Proposition 3. Since \( U \) is concave, if the risk premium is positive then \( U/M[x] \geq U/[h(\bar{w}_2)\bar{z}_2] \). It remains to show that
\[ \frac{M'[x]}{h(\bar{w}_2)} \geq 1 \] under non-increasing absolute risk aversion, which is what we proved above in Proposition 5.

The second sufficient condition for precautionary saving is non-increasing absolute risk aversion and \( s \geq 1 \). To prove this, write again the entire condition for precautionary saving
\[ U/[v^{-1}[E v[h(w_2)(w_2 + x)]]] \geq U/[h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2) \quad (B.12) \]

If absolute risk aversion is non-increasing we have \( E v/[h(w_2)(w_2 + x)] \geq 1 \). Furthermore, since \( \text{cov} \left( v/[h(w_2)(w_2 + x)], h(w_2) \right) \geq 0 \) it is sufficient to show that
\[
U^\prime \{v^{-1}\{Ev[h(w_2)(w_2 + x)]}\}\cdot Eh(w_2) \geq U^\prime [h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2) \quad (B.13)
\]

Risk aversion implies that \(v^{-1}\{Ev[h(w_2)(w_2 + x)]\} \leq Eh(w_2)(w_2 + x)\). Moreover, \(Eh(w_2)(w_2 + x) \leq Eh(w_2)E(w_2 + x)\), since \(cov((w_2 + x), h(w_2)) \leq 0\). Therefore,

\[
v^{-1}\{Ev[h(w_2)(w_2 + x)]\} \leq Eh(w_2)(w_2 + x) \leq Eh(w_2)E(w_2 + x) \quad (B.14)
\]

It is therefore sufficient to show that

\[
U^\prime \{Eh(w_2)\bar{z}_2\}Eh(w_2) \geq U^\prime [h(\bar{w}_2)\bar{z}_2]h(\bar{w}_2) \quad (B.15)
\]

Since \(Eh(w_2) \geq h(\bar{w}_2)\) this is equivalent to an increase in the rate of interest, i.e. the condition holds if \(s \geq 1\).

**B.4. Proof of Proposition 8**

The proof of Proposition 8 is identical to the case with exogenous income risk. See Appendix A.2, Eqs. A.17-A.20
REFERENCES


Table 1. Precautionary strength for different values of risk aversion and fluctuations aversion

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \rho )</th>
<th>Wage risk</th>
<th>Non-wage risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>20</td>
<td>1.2</td>
<td>3.2</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>6.0</td>
<td>21.0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.0</td>
<td>4.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>2.6</td>
<td>1.5</td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
<td>0.75</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Note: Utility is power Cobb-Douglas with \( a = b = 0.5 \)